

Non-intersecting Brownian walkers and Yang-Mills theory on the sphere

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Abstract

We study a system of N non-intersecting Brownian motions on a line segment $[0, L]$ with periodic, absorbing and reflecting boundary conditions. We show that the normalized reunion probabilities of these Brownian motions in the three models can be mapped to the partition function of two-dimensional continuum Yang-Mills theory on a sphere respectively with gauge groups $U(N)$, $Sp(2N)$ and $SO(2N)$. Consequently, we show that in each of these Brownian motion models, as one varies the system size L , a third order phase transition occurs at a critical value $L = L_c(N) \sim \sqrt{N}$ in the large N limit. Close to the critical point, the reunion probability, properly centered and scaled, is identical to the Tracy-Widom distribution describing the probability distribution of the largest eigenvalue of a random matrix. For the periodic case we obtain the Tracy-Widom distribution corresponding to the GUE random matrices, while for the absorbing and reflecting cases we get the Tracy-Widom distribution corresponding to GOE random matrices. In the absorbing case, the reunion probability is also identified as the maximal height of N non-intersecting Brownian excursions (“watermelons” with a wall) whose distribution in the asymptotic scaling limit is then described by GOE Tracy-Widom law. In addition, large deviation formulas for the maximum height are also computed.

1 Introduction

1.1 Background results

It is a well-known result that $U(N)$ lattice QCD in two dimensions with Wilson's action [54] exhibits a third order phase transition in the large N limit [23, 53]. This is shown by forming the partition function for the plaquettes, which factorizes as a product of partition functions for each individual plaquette. The latter is identified with a zero-dimensional unitary matrix model having partition function given by

$$G_N(b) := \left\langle e^{bN\text{Tr}(U+U^\dagger)} \right\rangle_{U \in U(N)}, \quad (1)$$

where the matrices $U \in U(N)$ are chosen with Haar measure and b is the scaled coupling.

The matrix integral (1) depends only on the N eigenvalues of U , and in terms of these variables it can be written

$$G_N(b) = \frac{1}{(2\pi)^N N!} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_N \prod_{l=1}^N e^{2bN \cos \theta_l} \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^2. \quad (2)$$

This can be interpreted as a partition function for a classical gas of charged particles, confined to the unit circle, and repelling via logarithmic pair potential $-(1/2) \log |e^{i\theta} - e^{i\phi}|$ at the inverse temperature $\beta = 2$. The charges are also subject to the extensive one-body potential $bN \cos \theta$. In the form (2) the $N \rightarrow \infty$ limit can be computed with the result [23]

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log G_N(b) = \begin{cases} b^2, & 0 < b < \frac{1}{2} \\ 2b - \frac{3}{4} - \frac{1}{2} \log 2b, & b > \frac{1}{2}, \end{cases} \quad (3)$$

which is indeed discontinuous in the third derivative at $b = 1/2$.

Some fifteen years after the work [23, 53] the same matrix integral (1) appeared in a completely different setting. Consider a unit square, and place points uniformly at random, with the number of points n chosen according to the Poisson distribution $P(n) = \frac{\lambda^{2n}}{n!} e^{-\lambda^2}$ with mean $\langle n \rangle = \lambda^2$. Starting at $(0, 0)$ and finishing at $(1, 1)$ form a piecewise linear path by joining dots with line segments of positive slope. There are evidently many such paths. From them, choose a *longest* path, *i.e.* a path with the maximum number of dots on it (see Fig. (1)). Let h^\square denote the length of this longest path. Clearly h^\square is a random variable that fluctuates from one configuration of points to another and its probability distribution has been studied extensively in the context of the directed last passage percolation model due to Hammersley (see e.g. [19, §10.9]). For the cumulative distribution

of h^\square , one gets [21, 45] (the first of the references gives a Toeplitz integral form equivalent to (2), while the second identifies the matrix integral explicitly)

$$\Pr(h^\square < N) = e^{-\lambda^2} \left\langle e^{\lambda \text{Tr}(U+U^\dagger)} \right\rangle_{U \in \text{U}(N)}. \quad (4)$$

In the limit of large λ , we set $\lambda = bN$, it follows from (2) that in the large N limit

$$\Pr(h^\square < N) = \begin{cases} 1, & 0 < b < \frac{1}{2} \\ e^{N^2(-b^2+2b-3/4-(1/2)\log 2b)}, & b > \frac{1}{2}, \end{cases} \quad (5)$$

hence providing for (3) the interpretation as a large deviation formula for a probabilistic quantity in a statistical model [27].

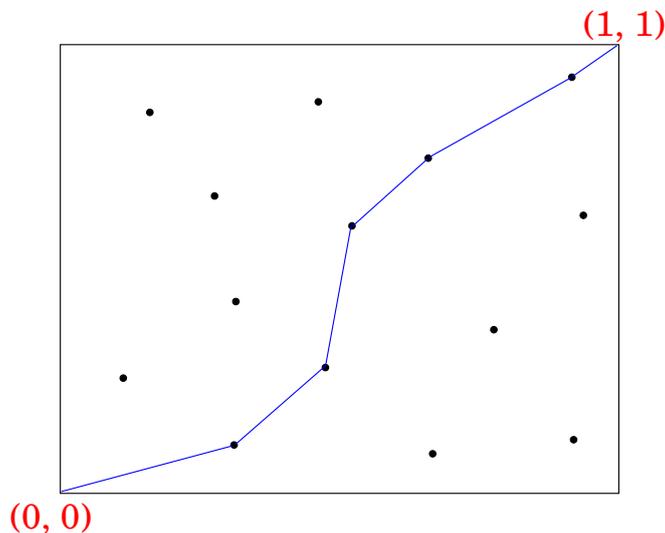


Figure 1: Points (black dots) are distributed uniformly in a unit square with mean density λ^2 . The length of any up-right path connecting the origin $(0,0)$ and the corner $(1,1)$ is defined by the number of points on the line. A path with the longest length is shown by the (blue) solid line.

The desire to relate 2d lattice QCD to string theory focussed attention on the so-called double scaling limit of matrix integrals. Here, in addition to $N \rightarrow \infty$ the coupling is tuned in the neighbourhood of the critical point $b = 1/2$ to give a well-defined scaling limit. It turns out that if one zooms in the neighbourhood of the critical point $b = 1/2$ and magnifies it by a factor $N^{2/3}$, i.e., one takes the limit $(1/2 - b) \rightarrow 0$, $N \rightarrow \infty$, but keeping the product $t = 2^{4/3}(1/2 - b)N^{2/3}$ fixed, then the second derivative of the free energy (the specific heat) tends to a function

of the single scaled variable t . In the case of the matrix integral (1) this double scaling limit was first analyzed by Periwal and Shevitz [42] whose result, using our notations, can be translated in the following form: setting $b = 1/2 - 2^{-4/3}N^{-2/3}t$

$$\frac{d^2}{dt^2} \lim_{N \rightarrow \infty} [\log G_N(b) - b^2 N^2] = -q^2(t), \quad (6)$$

where $q(t)$ satisfies the special case $\alpha = 0$ of the Painlevé II differential equation

$$u'' = 2u^3 + tu + \alpha. \quad (7)$$

In other words

$$q''(t) = 2q^3(t) + tq(t). \quad (8)$$

However no boundary condition was specified until Gross and Matytsin [22] refined the working of [42] to obtain a result which implies

$$q(t) \underset{t \rightarrow \infty}{\sim} \text{Ai}(t) \quad (9)$$

(see also the earlier reference [39] for identification of $\text{Ai}(t)$ in a similar context, and Section 3.2 below), where $\text{Ai}(t)$ denotes the Airy function.

In the context of the Hammersley model, recalling that in (4) $\lambda = bN$, we see that as a consequence of (6)

$$\lim_{N \rightarrow \infty} \Pr \left(\frac{h^\square - N}{(N/2)^{1/3}} < t \right) = \exp \left(- \int_t^\infty (s-t)q^2(s)ds \right). \quad (10)$$

This result was first obtained in the probability literature [3] independent of the working of [42]. In fact, the authors of [3] were interested in studying a ‘de-Poissonized’ version of the Hammersley model, where the number of dots in the unit square is a fixed number N , and not a Poisson distributed random variable. This ‘de-Poissonized’ Hammersley model is, in turn, related to the so-called Ulam problem where one studies the statistics of the length of the longest increasing subsequence of a random permutation of N integers (for a survey see [1, 34]). The length of the longest path in the ‘de-Poissonized’ Hammersley model has the same probability distribution as the length of the longest increasing subsequence in the random permutation. The Ulam problem has recently turned out to be a key model that connects various problems in combinatorics, physics and probability [1, 41, 34].

Remarkably, the right-hand side (rhs) of (10) admits a second interpretation within random matrix theory. Consider the Gaussian unitary ensemble (GUE) of the set of $N \times N$ complex Hermitian matrices X with measure proportional to $e^{-\text{Tr}X^2}$. Let λ_{\max} denote the largest eigenvalue. Its average, in the large N limit,

is simply $\langle \lambda_{\max} \rangle \simeq \sqrt{2N}$. The typical fluctuations of λ_{\max} around its average are very small of order $\sim N^{-1/6}$. It turns out that the probability distribution of these typical fluctuations have a limiting distribution [49]

$$\lim_{N \rightarrow \infty} \Pr \left(2^{1/2} N^{1/6} (\lambda_{\max} - \sqrt{2N}) < t \right) = \exp \left(- \int_t^{\infty} (s - t) q^2(s) ds \right) := \mathcal{F}_2(t), \quad (11)$$

known as the $\beta = 2$ Tracy-Widom distribution. This distribution function has recently appeared in a number of problems ranging from physics to biology [41, 34]. So (10) and (11) link distribution functions in two seemingly unrelated problems. In addition, the same Tracy-Widom function also appears in the scaled specific heat of the $U(N)$ lattice QCD in two dimensions as demonstrated by Eq. (6).

1.2 Statement of problems and summary of new results

In the previous subsection, we have seen that the partition function of a two-dimensional field theory model [$U(N)$ lattice QCD with Wilson action], when multiplied by a factor $e^{-\lambda^2}$ (see Eq. (4)), can be interpreted as the cumulative probability distribution of a certain random variable in the statistical physics/probability problem of Hammersley's directed last percolation model. It is then a natural question to ask if such connections can be established between other field theory models (on one side) and statistical physics models (on the other side). In this paper we establish another connection, namely between the continuum Yang-Mills gauge theory in two dimensions on a sphere (the field theory model) and the system of N non-intersecting Brownian motions on a line segment $[0, L]$ (the statistical physics model).

Non-intersecting random walkers, first introduced by de Gennes [14], followed by Fisher [18], have been studied extensively in statistical physics as they appear in a variety of physical contexts ranging from wetting and melting all the way to polymers and vortex lines in superconductors. Lattice versions of such walkers have also beautiful combinatorial properties [32]. Non-intersecting Brownian motions, defined in continuous space and time, have also recently appeared in a number of contexts, in particular its connection to the random matrix theory have been noted in a variety of situations [17, 28, 29, 40, 43, 47, 51]. In this paper we introduce three new models of non-intersecting Brownian motions and establish their close connections to the Yang-Mills gauge theory in two dimensions on a sphere.

Specifically, we consider a set of N non-intersecting Brownian motions on a finite segment $[0, L]$ of the real line with different boundary conditions. Assuming that all the walkers start from the vicinity of the origin, we then define the reunion probability as the probability that the walkers reunite at the origin after a fixed interval of time which can be set to unity without any loss of generality. Next

we ‘normalize’ this reunion probability in a precise way to be defined shortly. In one case, namely when both boundaries at 0 and L are absorbing, one can relate this ‘normalized’ reunion probability to the probability distribution of the maximal height of N non-intersecting Brownian excursions. We show in this paper how to map this normalized reunion probability in the Brownian motion models to the exactly solvable partition function (up to a multiplicative factor) of two-dimensional Yang-Mills theory on a sphere. The boundary conditions at the edges 0 and L select the gauge group of the associated Yang-Mills theory. We consider three different boundary conditions: periodic (model I), absorbing (model II) and reflecting (model III) which correspond respectively to the following gauge groups in the Yang-Mills theory: (I) periodic $\rightarrow U(N)$ (II) absorbing $\rightarrow Sp(2N)$ and (III) reflecting $\rightarrow SO(2N)$.

Using the known results on the partition function from the Yang-Mills theory, we will show how these normalized reunion probabilities in the Brownian motion models can be related to the limiting Tracy-Widom distribution of the largest eigenvalue in some particular random matrix ensembles. The latter, in addition to the distribution $\mathcal{F}_2(t)$ relating to complex Hermitian matrices, involves its companion $\mathcal{F}_1(t)$ for real symmetric matrices. Explicitly with the GOE specified as the set of $N \times N$ real symmetric matrices X with measure proportional to $e^{-\text{Tr}X^2/2}$, and λ_{\max} denoting the largest eigenvalue, one has [50]

$$\lim_{N \rightarrow \infty} \Pr\left(\sqrt{2}N^{\frac{1}{6}}(\lambda_{\max} - \sqrt{2N}) < t\right) = \exp\left(-\frac{1}{2} \int_t^\infty ((s-t)q^2(s) - q(s)) ds\right) \\ := \mathcal{F}_1(t). \quad (12)$$

We consider the following three models of N non-intersecting Brownian walkers on a one-dimensional line segment $[0, L]$ with different boundary conditions. Let the walkers be labelled by their positions at time τ , i.e., by $x_1(\tau) < \dots < x_N(\tau)$.

Model I: In the first model we consider periodic boundary conditions on the line segment $[0, L]$. Alternatively, one can think of the domain as a circle of circumference L (of radius $L/2\pi$). All walkers start initially in the vicinity of a point on the circle which we call the origin. We can label the positions of the walkers by their angles $\{\theta_1, \theta_2, \dots, \theta_N\}$ (see Appendix A for details). Let the initial angles be denoted by $\{\epsilon_1, \epsilon_2, \dots, \epsilon_N\}$ where ϵ_i ’s are small. Eventually we will take the limit $\epsilon_i \rightarrow 0$. We denote by $R_L^I(1)$ the reunion probability after time $\tau = 1$ (note that the walkers, in a bunch, may wind the circle multiple times), i.e, the probability that the walkers return to their initial positions after time $\tau = 1$ (staying non-intersecting over the time interval $\tau \in [0, 1]$). The superscript I corresponds to model I. Evidently $R_L^I(1)$ depends on N and also on the starting angles $\{\epsilon_1, \epsilon_2, \dots, \epsilon_N\}$. To avoid this additional dependence on the ϵ_i ’s, let us

introduce the normalized reunion probability defined as the ratio

$$\tilde{G}_N(L) = \frac{R_L^I(1)}{R_\infty^I(1)}, \quad (13)$$

where we assume that we have taken the $\epsilon_i \rightarrow 0$ limit. The ϵ dependence actually cancels out between the numerator and the denominator (see Appendix A) and hence $\tilde{G}_N(L)$ depends only on N and L . In Appendix A, we calculate $\tilde{G}_N(L)$ explicitly and show that

$$\tilde{G}_N(L) = \frac{A_N}{L^{N^2}} \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} \Delta^2(n_1, \dots, n_N) e^{-(2\pi^2/L^2) \sum_{j=1}^N n_j^2}, \quad (14)$$

where

$$\Delta(n_1, \dots, n_N) = \prod_{1 \leq i < j \leq N} (n_i - n_j),$$

and the prefactor

$$A_N = \frac{1}{(2\pi)^{N/2-N^2} \prod_{j=0}^{N-1} \Gamma(j+2)} \quad (15)$$

ensures that $\tilde{G}_N(L \rightarrow \infty) = 1$. In the next section we will see that this normalized reunion probability $\tilde{G}_N(L)$ is, up to a prefactor, exactly identical to the partition function of the 2-d Yang-Mills theory on a sphere with gauge group $U(N)$.

We remark that for the non-intersecting Brownian motions on a circle, a similar mapping was first noticed by Minahan and Polychronakos [38], with a slightly different normalization than ours. However, the behavior of the normalized reunion probability $\tilde{G}_N(L)$ as a function of the system size L was not analysed in [38] and consequently they did not uncover the existence of the Tracy-Widom distribution $\mathcal{F}_2(t)$ near the critical point $L_c(N) = 2\sqrt{N}$ for large N in the reunion probability, which is indeed one of our main findings in this paper.

Model II: In the second model the domain is the line segment $[0, L]$ with *absorbing boundary conditions at both boundaries 0 and L*. Once again, the N non-intersecting Brownian motions start initially at the positions, say, $\{\epsilon_1, \epsilon_2, \dots, \epsilon_N\}$ in the vicinity of the origin where eventually we will take the limit $\epsilon_i \rightarrow 0$ for all i . As in Model I, we define the reunion probability $R_L^{II}(1)$ as the probability that the walkers return to their initial positions after a fixed time $\tau = 1$ (staying non-intersecting over the time interval $\tau \in [0, 1]$). Analogous to Model I, we define the normalized reunion probability

$$\tilde{F}_N(L) = \frac{R_L^{II}(1)}{R_\infty^{II}(1)}, \quad (16)$$

which becomes independent of the starting positions ϵ_i 's in the limit when $\epsilon_i \rightarrow 0$ for all i . Hence, $\tilde{F}_N(L)$ depends only on N and L .

This ratio $\tilde{F}_N(L)$ in Model II also has a different probabilistic interpretation. Consider the same model but now on the semi-infinite line $[0, \infty]$ with still absorbing boundary condition at 0. The walkers, as usual, start in the vicinity of the origin and are conditioned to return to the origin exactly at $\tau = 1$ (see Fig. (2)). If one plots the space-time trajectories of the walkers, a typical configuration looks like half of a watermelon (see Fig. (2)), or a watermelon in presence of an absorbing wall. Such configurations of Brownian motions are known as non-intersecting Brownian excursions and their statistical properties have been studied quite extensively in the recent past. A particular observable that has generated some recent interests is the so-called 'height' of the watermelon [6, 16, 20, 30, 47, 31] defined as follows (see also Ref. [8] for a related quantity in the context of Dyson's Brownian motion). Let H_N denote the maximal displacement of the rightmost walker x_N in this time interval $\tau \in [0, 1]$, i.e., the maximal height of the topmost path in half-watermelon configuration (see Fig. (2)), i.e., $H_N = \max_{\tau} \{x_N(\tau), 0 < \tau < 1\}$. This global maximal height H_N is a random variable which fluctuates from one configuration of half-watermelon to another. What is the probability distribution of H_N ? For $N = 1$ the distribution of H_N is easy to compute and already for $N = 2$ it is somewhat complicated [30]. Recently, however, an exact formula for the distribution of H_N , valid for all N , was derived in [47] using Fermionic path integral method.

To relate the distribution of H_N in the semi-infinite system defined above to the ratio of reunion probabilities in the finite segment $[0, L]$ defined in (16), it is useful to consider the cumulative probability $\Pr(H_N \leq L)$ in the semi-infinite geometry, where L now is just a variable. To compute this cumulative probability, we need to calculate the fraction of half-watermelon configurations (out of all possible half-watermelon configurations) that never cross the level L , i.e., whose heights stay below L over the time interval $\tau \in [0, 1]$ (see Fig. (2)). This fraction can be computed by putting an absorbing boundary at L (thus killing all configurations that touch/cross the level L). It is then clear that $\Pr(H_N \leq L)$ is nothing but the normalized reunion probability $\tilde{F}_N(L)$ defined in (16). As mentioned above, this cumulative probability distribution of the maximal height was computed exactly in Ref. [47] (see also [16], [31] for related computations)

$$\begin{aligned} \tilde{F}_N(L) &:= \Pr(H_N \leq L) \\ &= \frac{B_N}{L^{2N^2+N}} \sum_{n_1=1}^{\infty} \cdots \sum_{n_N=1}^{\infty} \Delta^2(n_1^2, \dots, n_N^2) \left(\prod_{j=1}^N n_j^2 \right) e^{-\frac{\pi^2}{2L^2} \sum_{j=1}^N n_j^2}, \end{aligned} \quad (17)$$

where $B_N = \pi^{2N^2+N} / (2^{N^2-N/2} \prod_{j=0}^{N-1} \Gamma(2+j)\Gamma(3/2+j))$. A brief derivation of this result is given in Appendix B. Note that a similar probabilistic interpretation

(i.e. as a cumulative distribution of a random variable) does not exist for the normalized reunion probability $\tilde{G}_N(L)$ in Model I. This is evident from the fact that, unlike $\tilde{F}_N(L)$ in Model II, the quantity $\tilde{G}_N(L)$ in Model I is not bounded from above by 1 (this can even be seen by a direct computation in the $N = 1$ case, see (102) in Appendix A).

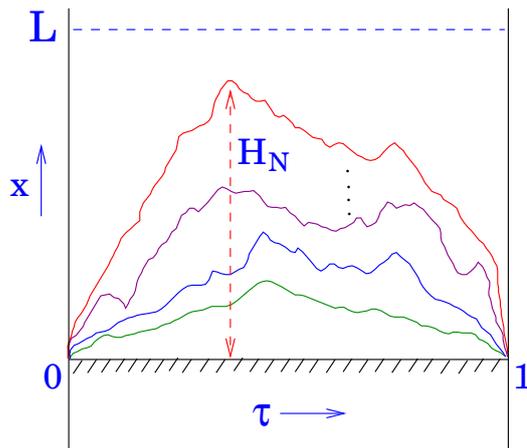


Figure 2: Trajectories of N non-intersecting Brownian motions $x_1(\tau) < x_2(\tau) < \dots < x_N(\tau)$, all start at the origin and return to the origin at $\tau = 1$, staying positive in between. $\tilde{F}_N(L)$ denotes the probability that the maximal height $H_N = \max_{\tau} \{x_N(\tau), 0 \leq \tau \leq 1\}$ stays below the level L over the time interval $0 \leq \tau \leq 1$.

In the next section, we will see that the normalized reunion probability (or equivalently the maximal height distribution) $\tilde{F}_N(L)$ in Model II is, again up to a prefactor, precisely equal to the partition function of the 2-d Yang-Mills theory on a sphere, but with a gauge group $\text{Sp}(2N)$ (as opposed to the group $\text{U}(N)$ in Model I).

Model III: We consider a third model of non-intersecting Brownian motions where the walkers move again on a finite line segment $[0, L]$, but this time with *reflecting* boundary conditions at both boundaries 0 and L . Again the walkers start in the vicinity of the origin at time $\tau = 0$ and we consider the reunion probability $R_L^{III}(1)$ that they reunite at time $\tau = 1$ at the origin. Following Models I and II, we define

the normalized reunion probability

$$\tilde{E}_N(L) = \frac{R_L^{III}(1)}{R_\infty^{III}(1)}, \quad (18)$$

that is independent of the starting positions $\{\epsilon_1, \epsilon_2, \dots, \epsilon_N\}$ in the limit when all the ϵ_i 's tend to zero and hence depends only on N and L . Following similar steps as in Models I and II, but with reflecting boundary conditions at both 0 and L , we find the exact expression (see Appendix B)

$$\tilde{E}_N(L) = \frac{C_N}{L^{2N^2-N}} \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} \Delta^2(n_1^2, \dots, n_N^2) e^{-(\pi^2/L^2) \sum_{j=1}^N n_j^2}, \quad (19)$$

and the prefactor

$$C_N = \frac{\pi^{2N^2-N} 2^{3N/2-N^2}}{\prod_{j=0}^{N-1} \Gamma(2+j) \Gamma(1/2+j)} \quad (20)$$

ensures that $\tilde{E}_N(L \rightarrow \infty) = 1$. In the next section we will see that $\tilde{E}_N(L)$, up to a prefactor, is exactly identical to the partition function of the 2-d Yang-Mills theory on a sphere with gauge group $\text{SO}(2N)$.

Thus, we find that by changing the boundary conditions at the edges of the line segment $[0, L]$ in the non-intersecting Brownian motion models we can relate the normalized reunion probability to the partition function of the 2-d Yang-Mills theory on a sphere with an appropriate gauge group. Model I, II and III correspond respectively to gauge groups $\text{U}(N)$, $\text{Sp}(2N)$ and $\text{SO}(2N)$.

Summary of new results: Let us then summarize the main new results in this paper:

- We have shown that the normalized reunion probability of a set of N non-intersecting Brownian motions moving on a line segment $[0, L]$ with a prescribed boundary condition is identical, up to a prefactor, to the partition function of the 2-d Yang-Mills theory on a sphere with an appropriate gauge group which depends on the boundary conditions in the Brownian motion model. We have shown that three different boundary conditions lead respectively to the gauge groups: (I) periodic $\rightarrow \text{U}(N)$ (II) absorbing $\rightarrow \text{Sp}(2N)$ and (III) reflecting $\rightarrow \text{SO}(2N)$.

- The partition function of the 2-d Yang-Mills theory with a given gauge group is exactly solvable and many results are known. In particular, it is known that in the so-called ‘double scaling’ limit, the singular part of the Yang-Mills free energy satisfies a Painlevé equation. To use these results for the Brownian motion model via the correspondence established above, we need to however treat the prefactor

of the correspondence properly in the double scaling limit. Taking into account new terms arising via these prefactors, we show that the normalized reunion probabilities, appropriately centered and scaled as a function L for large but fixed N , also share a ‘double scaling’ limit where they are precisely described by the Tracy-Widom distribution. In the periodic case, one gets $\mathcal{F}_2(t)$ while for the absorbing and reflecting cases one gets $\mathcal{F}_1(t)$. Thus, this relates for the first time (to our knowledge) the Painlevé equation that appears in the Yang-Mills free energy to the Painlevé equation in the Tracy-Widom distribution.

- As a byproduct of this mapping, we also show that the 3-rd order phase transition between the strong and the weak coupling phases (separated by the double scaling regime) in the 2-d Yang-Mills theory translates into a 3-rd order phase transition in the behavior of the normalized reunion probability in the Brownian motion model, as one varies the system size L across a critical value $L_c(N) \sim \sqrt{N}$ for large but fixed N . The strong and weak coupling phases correspond respectively to the left and right large deviation (away from the critical value $L_c(N)$) behaviors of the normalized reunion probability as a function of L . Again using results from the Yang-Mills theory (taking into account the prefactors correctly), we thus obtain precise large deviation behaviors of these reunion probabilities. In particular, in the right large deviation behavior, we find a new type of crossover phenomenon.

- For the special case of absorbing boundary condition (Model II), this gives us a direct proof that the distribution of the maximal height H_N of a set of N non-intersecting Brownian excursions, when properly centered and scaled, has the Tracy-Widom distribution $\mathcal{F}_1(t)$ described in (12). This was shown before rather indirectly in Ref. [28] via a mapping to a polynuclear growth model. Here we obtain a direct proof of this result.

Let us remark that while some aspects of the analogies between non-intersecting Brownian paths and Yang-Mills theory on the sphere have been noticed in earlier publications [26, 25], this precise correspondence between the normalized reunion probability in the Brownian motion models (with different boundary conditions) and the partition function of the 2-d Yang-Mills theory on a sphere (with different gauge groups) seems to be new, to the best of our knowledge (except for the periodic case when a similar correspondence was noted in [38]). More importantly, the probabilistic connection between the Yang-Mills partition function in the double scaling limit and the Tracy-Widom distribution of the largest eigenvalue of a random matrix (established here using the connection via non-intersecting Brownian motions), to our knowledge, was not noticed earlier.

The rest of the paper is organised as follows. In Section 2, we briefly recapitulate the exact solution of the continuum Yang-Mills theory in two dimensions and then establish the correspondence between the partition functions of the gauge theory with normalized reunion probabilities in the Brownian motion models defined above. Next we study the consequences of this correspondence for Model I and Model II respectively in Sections 3 and 4. In particular, we will see how the Tracy-Widom distributions $\mathcal{F}_2(t)$ and $\mathcal{F}_1(t)$ emerge in the double scaling limit in Models I and II respectively. For Model II, this correspondence also provides us with the precise asymptotic distribution of the maximal height for N non-intersecting Brownian excursions. We do not discuss Model III in details here as the analysis is very similar to that of Model II. The detailed derivation of the expression of $\tilde{G}_N(L)$ in (14) is provided in the Appendix A. The derivations of $\tilde{F}_N(L)$ in (17) and of $\tilde{E}_N(L)$ in (19) follow via similar calculations which are briefly outlined in Appendix B.

2 Correspondence between 2-d Yang-Mills theory and non-intersecting Brownian motions on a line segment $[0, L]$

We start by briefly recapitulating how one computes the 2-d Yang-Mills partition function [37, 46, 24] (see also the review by Cordes, Moore and Ramgoolam [9]). Consider a general orientable two-dimensional manifold \mathcal{M} with corresponding volume form \sqrt{g} . At each point x of this manifold sits a gauge field $A_\mu(x)$ (with the space index $\mu = 1, 2$) which is an $(N \times N)$ matrix. For simplicity, let us first consider pure $U(N)$ Yang-Mills gauge theory whose partition function in continuum is defined by the functional integral

$$\mathcal{Z}_{\mathcal{M}} = \int [\mathcal{D}A_\mu] e^{-(1/4\lambda^2) \int_{\mathcal{M}} \sqrt{g} \text{Tr}[F^{\mu\nu} F_{\mu\nu}] d^2x}, \quad (21)$$

where λ is a coupling constant and the field strengths are defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]. \quad (22)$$

Under a local gauge transformation $A_\mu \rightarrow S^{-1}(x)A_\mu S(x) - iS^{-1}(x)\partial_\mu S(x)$ (where $S(x)$ is an $(N \times N)$ unitary matrix), the field strengths transform as $F_{\mu\nu} \rightarrow S^{-1}(x)F_{\mu\nu}S(x)$ thus keeping the action in (21) gauge invariant.

The partition function in (21) in two dimensions can be computed exactly via the original idea due to Migdal [37]. One can actually use a particular lattice regularization of the continuum theory which is both exact and additive in the

following sense [24]. One can divide the manifold into polygons (for example one can choose triangles as basic units or plaquettes) and define a unitary matrix U_L sitting at the center of each link L of this triangulated manifold (see Fig. 3). Then the lattice regularized partition function can be written [24]

$$\mathcal{Z}_{\mathcal{M}}(\text{lattice}) = \int \prod_L dU_L \prod_{\text{plaquettes}} Z_P[U_P], \quad (23)$$

where $U_P = \prod_{L \in \text{plaquette}} U_L$ is called the loop product introduced by Wilson [54] and $Z_P(U_P)$ is some appropriate lattice action associated with the plaquette P . The only constraint on the choice of Z_P is that it should reduce to the continuum action when the plaquette size goes to zero.

A standard choice for $Z_P(U_P)$ is the Wilson action [54]

$$Z_P(U_P) = \exp \left[b N \text{Tr}(U_P + U_P^\dagger) \right], \quad (24)$$

which does reduce to the continuum action in the limit when the plaquette area goes to zero. With this choice, the lattice partition function in (23) is exactly solvable [23, 53] as it reduces to computing a single matrix integral in Eq. (1) already discussed in the introduction. However, the Wilson action is not invariant under renormalization in the following sense. Following Migdal [37], one can take two adjacent plaquettes P_1 and P_2 with their respective actions Z_{P_1} and Z_{P_2} and fuses them to form a bigger plaquette with area equal to that of $P_1 + P_2$, after integrating out the unitary matrix sitting on the common link between the two plaquettes (see Fig. 3). This gives the Migdal recursion relation

$$\int dU_3 Z_{P_1}(U_1 U_2 U_3) Z_{P_2}(U_4 U_5 U_3^\dagger) = Z'_{P_1+P_2}(U_1 U_2 U_4 U_5). \quad (25)$$

In general, the renormalized plaquette action Z'_P does not have the same functional form as the bare action Z_P (which is indeed the case when one chooses Wilson action as the bare action). So, the natural thing to look for is the fixed point solution of this recursion relation that keeps the form of Z_P invariant under renormalization. Indeed, Migdal [37] and later Rusakov [46] showed that the appropriate fixed point action is given by the so-called heat-kernel action

$$Z_P = \sum_R d_R \chi_R(U_P) \exp \left[-\frac{\tilde{\lambda} A_P}{2N} C_2(R) \right], \quad (26)$$

where the sum runs over all irreducible representations R of $U(N)$, d_R is the dimension of R , $\chi_R(U_P)$ is the character of U_P in the representation R , $C_2(R)$ is

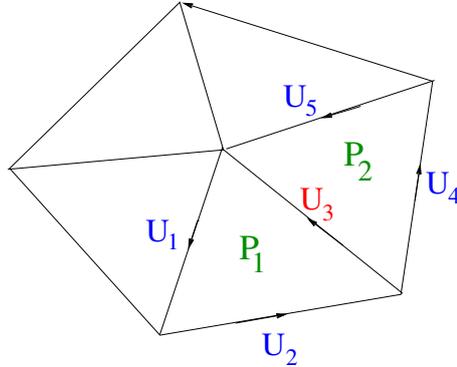


Figure 3: a typical triangulation of the two-dimensional manifold with unitary matrices U_L 's on the edges. One can fuse two triangles P_1 and P_2 by integrating the matrix U_3 along their common edge and obtain a parallelogram with four edges with matrices U_1 , U_2 , U_4 and U_5 on these edges.

the quadratic Casimir operator of R and A_P is the area of the plaquette. The coupling constant $\tilde{\lambda}$ is fixed and will henceforth be chosen to be unity: $\tilde{\lambda} = 1$. One can also verify that this heat-kernel action reduces to the continuum action in the limit when the area of each basic plaquette goes to zero [37, 46]. Note that this fixed point choice of Z_P makes the lattice representation of the continuum theory exact in the sense that the final result is independent of the triangulation as one can add as many triangles (and in whichever way) to cover the manifold thus approaching the continuum limit [24]. The name heat-kernel stems from the fact that one can express the heat-kernel action $Z_P(U) = \langle U | \exp[-(1/2N)A\Delta] | 1 \rangle$ where Δ is the Laplacian on the group [36]. This fact already gives a hint that there might be an underlying diffusion process inbuilt in this effective action, though the precise fact that they correspond to non-intersecting Brownian motions is still not evident at this point. Using the heat-kernel action Z_P in (26) one can then evaluate the full partition function on the manifold [46]

$$\mathcal{Z}_{\mathcal{M}} = \sum_R d_R^{2-2p} \exp \left[-\frac{A}{2N} C_2(R) \right], \quad (27)$$

where p is the genus of the manifold and A is the total surface area. This then

represents an exact solution of the partition function of the continuum Yang-Mills theory on a twodimensional manifold. Note that even though we have specifically used the group $U(N)$ for the discussion above, the result (27) is valid as well for other groups such as $SU(N)$, $Sp(N)$ and $SO(N)$.

For a sphere, using $p = 0$, one gets

$$\mathcal{Z}_{\mathcal{M}} = \sum_R d_R^2 \exp \left[-\frac{A}{2N} C_2(R) \right]. \quad (28)$$

The irreducible representations can be labelled by the lengths of the Young diagrams and one can explicitly express the partition function as a multiple sum. For the group $U(N)$, the partition function reads [24, 22]

$$\mathcal{Z}_{\mathcal{M}} := \mathcal{Z}(A; U(N)) = c_N e^{-A \frac{N^2-1}{24}} \sum_{n_1, \dots, n_N = -\infty}^{\infty} \Delta^2(n_1, \dots, n_N) e^{-(A/2N) \sum_{j=1}^N n_j^2}, \quad (29)$$

where $\Delta(n_1, n_2, \dots, n_N)$ is the van der Monde determinant defined in (1.2) and c_N is a constant independent of A . For the groups $Sp(2N)$ and $SO(2N)$ one can similarly express the partition function as a multiple sum [11]. For example, for $Sp(2N)$ one gets

$$\mathcal{Z}(A; Sp(2N)) = \hat{c}_N e^{A(N+\frac{1}{2})\frac{N+1}{12}} \sum_{n_1, \dots, n_N = -\infty}^{\infty} \Delta^2(n_1^2, \dots, n_N^2) \left(\prod_{j=1}^N n_j^2 \right) e^{-\frac{A}{4N} \sum_{j=1}^N n_j^2}, \quad (30)$$

where \hat{c}_N is independent of A . Similarly, for the $SO(2N)$ group, one obtains [11]

$$\mathcal{Z}(A; SO(2N)) = \hat{b}_N e^{A(N-\frac{1}{2})\frac{N-1}{12}} \sum_{n_1, \dots, n_N = -\infty}^{\infty} \Delta^2(n_1^2, \dots, n_N^2) e^{-\frac{A}{4N} \sum_{j=1}^N n_j^2}, \quad (31)$$

where \hat{b}_N is independent of A .

Comparing the formulas in (14), (17) and (19) with the expression for partition functions respectively in (29), (30), and (31), we see that the normalized reunion probabilities in the three models of non-intersecting Brownian motions, up to prefactors that can be computed explicitly, are isomorphic to the partition functions of the Yang-Mills theory on a sphere with respective gauge groups $U(N)$, $Sp(2N)$ and $SO(2N)$, provided we make the identification $A/2N \rightarrow 2\pi^2/L^2$ in Model I, $A/4N \rightarrow \pi^2/2L^2$ in model II and $A/4N \rightarrow \pi^2/L^2$ in model III. More precisely, up

to known prefactors,

$$\tilde{G}_N(L) \propto \mathcal{Z} \left(A = \frac{4\pi^2 N}{L^2}; \text{U}(N) \right), \quad (32)$$

$$\tilde{F}_N(L) \propto \mathcal{Z} \left(A = \frac{2\pi^2 N}{L^2}; \text{Sp}(2N) \right), \quad (33)$$

$$\tilde{E}_N(L) \propto \mathcal{Z} \left(A = \frac{4\pi^2 N}{L^2}; \text{SO}(2N) \right). \quad (34)$$

This correspondence between the normalized reunion probability for non-intersecting Brownian motions with different boundary conditions and the Yang-Mills partition functions on a sphere with corresponding gauge groups is one of the main observations of this paper.

In the next two sections we study the consequences of this correspondence in detail for Model I and II. We skip detailed studies of Model III since it can be handled exactly in the same way as Model II. The main point is to derive the precise asymptotics of the normalized reunion probabilities in the two models (in particular for Model II this will give us the asymptotic behavior of the distribution of the maximal height H_N for non-intersecting Brownian excursions) by using the known behavior of the asymptotic properties of the partition functions of the corresponding gauge theory, albeit taking into account correctly the L and N dependence of the respective prefactors.

3 Brownian walkers on a circle: Model I

Comparing (29) and (14) we have the following exact identity between the normalized reunion probability and the $\text{U}(N)$ Yang-Mills partition function on a sphere (upon substituting $A = 4\pi^2 N/L^2$ in (29))

$$\tilde{G}_N(L) = \frac{A_N}{c_N} e^{\pi^2 N(N^2-1)/6L^2} L^{-N^2} \mathcal{Z} \left(\frac{4\pi^2 N}{L^2}; \text{U}(N) \right), \quad (35)$$

where the constants A_N , given in (15), is independent of L . Similarly the constant c_N is independent of L and can be fixed in any way. Later, we choose c_N such that for large N , $\log c_N \simeq -N^2 \log N$. This choice of c_N ensures that the free energy $\log \mathcal{Z}_M \sim \mathcal{O}(N^2)$ for large N (see below).

In Ref. [15] it was shown that for large N , $\mathcal{Z}(A; \text{U}(N))$ exhibits a 3-rd order phase transition at the critical value $A = \pi^2$ separating a weak coupling regime for $A < \pi^2$ and a strong coupling regime for $A > \pi^2$ (see Fig. 4). This is the so called Douglas-Kazakov phase transition [15]. Using the correspondence $L^2 := 4\pi^2 N/A$, we then find that $\tilde{G}_N(L)$, considered as a function of L with N

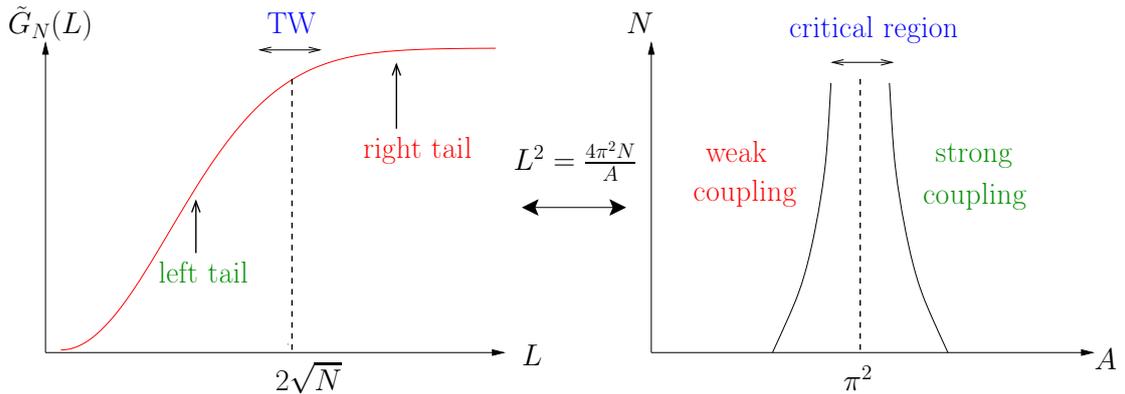


Figure 4: **Left:** Schematic sketch of $\tilde{G}_N(L)$ as defined in Eq. (13) for N vicious walkers on a circle, as a function of L , for fixed but large N . **Right:** Sketch of the phase diagram in the plane (A, N) of two-dimensional Yang-Mills theory on a sphere with the gauge group $U(N)$ as obtained in Ref. [15]. The weak (strong) coupling phase in the right panel corresponds to the right (left) tail of $\tilde{G}_N(L)$ in the left panel. The critical region around $A = \pi^2$ in the right panel corresponds to the Tracy-Widom (TW) regime in the left panel around the critical point $L_c(N) = 2\sqrt{N}$.

large but fixed, must also exhibit a 3-rd order phase transition at the critical value $L_c(N) = 2\sqrt{N}$. Furthermore, the weak coupling regime ($A < \pi^2$) corresponds to $L > 2\sqrt{N}$ and thus describes the right tail of $\tilde{G}_N(L)$, while the strong coupling regime corresponds to $L < 2\sqrt{N}$ and describes instead the left tail of $\tilde{G}_N(L)$ (see Fig. 4). The critical regime around $A = \pi^2$ is the so-called "double scaling limit" in the matrix model and has width of order $N^{-2/3}$. It corresponds, as we will see, to the region (of width $N^{-1/6}$ around $L = 2\sqrt{N}$) where $\tilde{G}_N(L)$, correctly shifted and scaled, is described by the Tracy-Widom distribution $\mathcal{F}_2(t)$.

3.1 Large deviation tails

Let us first consider the behavior of $\tilde{G}_N(L)$, as a function of L , in the two tails far away from the critical value $L_c(N) = 2\sqrt{N}$ for large but fixed N . To do this, we can exploit the exact identity in (35) and use the known properties of the $U(N)$ partition function respectively in the strong ($A > \pi^2$) and the weak ($A < \pi^2$) coupling phases. The left ($L < 2\sqrt{N}$) and the right ($L > 2\sqrt{N}$) tails of $\tilde{G}_N(L)$, away from the critical point $L_c(N) = 2\sqrt{N}$, correspond respectively to the behavior of the partition function in the strong ($A > \pi^2$) and weak ($A < \pi^2$) coupling phases.

Let us begin by summarizing the known properties of the $U(N)$ partition function in the strong and weak coupling phases. The summand in (29) can naturally be regarded as a function of n_i/N ($i = 1, \dots, N$). In the large N limit the variables $\{n_i/N\}_{i=1, \dots, N}$ approximate the coordinates of a continuous N particle system. Associated with the particles is a density $\rho(x)$, but because the lattice spacing is $1/N$ and there are N particles the density at any point cannot exceed 1, and thus $\rho(x) \leq 1$ for all x . By using this viewpoint to perform a constrained saddle point analysis of (29) (see also Section 4.1 below), it was shown by Douglas and Kazakov [15] that

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log \mathcal{Z}(A; U(N)) = \begin{cases} F_-(A), & A \leq \pi^2, \\ F_+(A), & A \geq \pi^2, \end{cases} \quad (36)$$

where

$$F_-(A) = -\frac{3}{4} - \frac{A}{24} - \frac{1}{2} \log A. \quad (37)$$

The function $F_+(A)$ about $A = \pi^2$ agrees with $F_-(A)$ up to $\mathcal{O}((A - \pi^2)^2)$ in its power series expansion. Thus [15, eq. (40)]

$$F_+(A) - F_-(A) = \frac{1}{3\pi^6} (A - \pi^2)^3 + \mathcal{O}((A - \pi^2)^4). \quad (38)$$

Its explicit form — or more precisely that of its derivative — is given in terms of elliptic integrals. Specifically, with $K \equiv K(k)$, $E \equiv E(k)$ denoting standard elliptic integrals, where k is specified by the requirement

$$\frac{A}{4} = (2E - k'^2 K)K, \quad k'^2 = 1 - k^2, \quad (39)$$

and

$$a = \frac{4K}{A}, \quad (40)$$

one has [15, eq. (35)]

$$F'_+(A) = \frac{a^2}{6} - \frac{a^2 k'^2}{12} - \frac{1}{24} + \frac{a^4 k'^4 A}{96}. \quad (41)$$

In (36), the case $A < \pi^2$ corresponds to the weak coupling phase and the constraint on the density of the variables n_i/N being less than 1 can be ignored, implying that the Riemann sum is well approximated by the corresponding multi-dimensional integral. In the latter the A dependence can be determined by scaling, and the resulting integral evaluated explicitly (see e.g. [19, eq. (4.140)]) to deduce the stated result. However for $A > \pi^2$ doing this would imply that the density is greater than 1, so the discrete sum is no longer well approximated by the continuous integral. The density saturates at 1 for some range of values of the variables n_i/N

about the origin, but goes continuously to zero to be supported on a finite interval (see e.g. Fig. 2 in [2]).

Let us now use these known results on the partition function in our exact identity (35) to derive the corresponding large deviation properties of $\tilde{G}_N(L)$. Setting, for convenience, $L = 2\sqrt{N}r$ in (35) so that $r = 1$ corresponds to the critical point, we get

$$\tilde{G}_N(2\sqrt{N}r) = \frac{A_N e^{\pi^2(N^2-1)/24r^2}}{c_N(2\sqrt{N}r)^{N^2}} \mathcal{Z}\left(\frac{\pi^2}{r^2}; U(N)\right). \quad (42)$$

We first note that A_N in Eq. (15) contains the product $\prod_{j=0}^{N-1} \Gamma(j+2) = G(N+2)$ in the denominator, where $G(x)$ denotes the Barnes G -function whose asymptotic expansion is (see e.g., [19, eq.(4.184)])

$$\ln(G(z+1)) = \frac{z^2 \ln z}{2} - \frac{3z^2}{4} + \frac{\ln(2\pi)}{2} z - \frac{1}{12} \ln(z) + \zeta'(-1) + \mathcal{O}(1/z). \quad (43)$$

Using this result and the choice $c_N \simeq e^{-N^2 \log N}$, we see that the leading $N^2 \log N$ term cancels when one considers the ratio

$$\frac{A_N e^{\pi^2(N^2-1)/24r^2}}{c_N(2\sqrt{N}r)^{N^2}} \sim e^{N^2(3/4 - \log(r/\pi) + \pi^2/24r^2) + \mathcal{O}(N)}. \quad (44)$$

Consequently, the result in Eq. (36) gives the large deviation formula

$$\tilde{G}_N(2\sqrt{N}r) \sim \begin{cases} 1, & r \geq 1 \\ e^{-N^2(F_-(\pi^2/r^2) - F_+(\pi^2/r^2))}, & r \leq 1. \end{cases} \quad (45)$$

This is then a new result on the far tails of the normalized reunion probability $\tilde{G}_N(L)$, as a function of L , for fixed but large N .

Note that the precise meaning of \sim in (45) is the following

$$-\lim_{N \rightarrow \infty} \frac{1}{N^2} \ln \left[\tilde{G}_N(2\sqrt{N}r) \right] = \begin{cases} 0, & r \geq 1 \\ F_-(\pi^2/r^2) - F_+(\pi^2/r^2), & r \leq 1. \end{cases} \quad (46)$$

Thus this calculation only gives the leading $\mathcal{O}(N^2)$ term. While in the left tail ($r \leq 1$) the leading term of $\mathcal{O}(N^2)$ is a finite nontrivial quantity, this leading $\mathcal{O}(N^2)$ term actually vanishes in the right tail ($r \geq 1$). So, to obtain the finer behavior in the right tail one needs to keep track of the next subleading correction of $\mathcal{O}(N)$ term in $\ln[\tilde{G}_N(2\sqrt{N}r)]$ for $r \geq 1$.

In the gauge theory context, this means that we need to obtain in the weak coupling phase, not only the leading term of $\mathcal{O}(N^2)$ but also the subleading corrections. Fortunately, these subleading corrections in the free energy $\ln \mathcal{Z}$ in the

weak coupling phase were also calculated by Gross and Matytsin [22, eq. (2.41)]

$$\begin{aligned} \log \mathcal{Z}(A; U(N)) &\sim N^2 F_-(A) + \frac{A}{24} - \frac{(-1)^N}{\sqrt{2\pi N}} \frac{A}{2\pi^2} \left(1 - \frac{A}{\pi^2}\right)^{-1/4} e^{-\frac{2\pi^2 N}{A} \gamma\left(\frac{A}{\pi^2}\right)} \\ &+ \mathcal{O}\left(e^{-\frac{4\pi^2 N}{A} \gamma\left(\frac{A}{\pi^2}\right)}\right), \end{aligned} \quad (47)$$

up to terms independent of A (these will depend on the precise form of c_N in (29)) where

$$\gamma(x) = \sqrt{1-x} - \frac{x}{2} \log \frac{1 + \sqrt{1-x}}{1 - \sqrt{1-x}} = \frac{2}{3}(1-x)^{3/2} + \mathcal{O}((1-x)^{5/2}). \quad (48)$$

We can then use this result in our identity (42) to predict the following right large deviation tail

$$1 - \tilde{G}_N(2\sqrt{N}r) \sim (-1)^N e^{-2Nr^2 \gamma(1/r^2)}, \quad r > 1, \quad (49)$$

which shows an interesting oscillatory behavior. Again this is a new result for the normalized reunion probability of the non-intersecting Brownian motions on a circle. Let us recall that $\tilde{G}_N(L)$ does not have the meaning of a cumulative distribution and hence the oscillating sign of $[1 - \tilde{G}_N(2\sqrt{N}r)]$ (with N) is not really problematic.

3.2 Double scaling limit

Having obtained the precise large N asymptotic behavior of $\tilde{G}_N(L)$ as a function of L in the left ($L < 2\sqrt{N}$) and the right ($L > 2\sqrt{N}$) tails, we now turn our attention to the behavior of $\tilde{G}_N(L)$ in the vicinity of the critical point, i.e., when L is close to $L_c(N) = 2\sqrt{N}$. In the gauge theory, this corresponds to the double scaling regime near the critical point $A = \pi^2$. Below, we first discuss the known results on the partition function in the double scaling regime. We then use these results in our exact identity (35) to show that correspondingly $\tilde{G}_N(L)$, in a narrow region $|L - 2\sqrt{N}| \sim N^{-1/6}$ around the critical point $L_c(N) = 2\sqrt{N}$, has the scaling behavior

$$\tilde{G}_N(L) \rightarrow \mathcal{F}_2\left(2^{2/3} N^{1/6} |L - 2\sqrt{N}|\right), \quad (50)$$

where the scaling function $\mathcal{F}_2(t)$ is precisely the Tracy-Widom distribution function for GUE random matrices defined by

$$\mathcal{F}_2(t) = \exp\left(-\int_t^\infty (s-t)q^2(s)ds\right), \quad (51)$$

with $q(s)$ satisfying the Painlevé equation (8). This is the main result of this subsection and details are provided below.

Gross and Matytsin [22] used the method of orthogonal polynomials (see e.g. [19, Ch. 5]) to analyse the $N \rightarrow \infty$ asymptotic behaviour of (29) when the coupling A is tuned in the neighbourhood of the transition point $A = \pi^2$. Explicitly, in what is referred to as the double scaling limit, this is achieved by taking $N \rightarrow \infty$ while keeping $(\pi^2 - A)N^{2/3}$ fixed.

As most clearly set out in [11, eq. (23)], for certain orthogonal polynomial normalizations $R_N^{(\pm)}$, the method of orthogonal polynomials gives

$$\frac{d^2}{dA^2} \log \mathcal{Z}(A; \text{U}(N)) = \frac{1}{4N^2} R_N^{(\pm)} (R_{N+1}^{(\pm)} + R_{N-1}^{(\pm)}) \quad (52)$$

with $+$ ($-$) chosen according to N being even (odd). Moreover, it is shown in [22] that for j near N

$$R_j^{(\pm)} = \frac{N^2}{\pi^2} \mp (-1)^j N^{5/3} f_1(x) + \mathcal{O}(N^{4/3}) \quad (53)$$

where

$$x = N^{2/3} \left(1 - \frac{A}{\pi^2}\right). \quad (54)$$

In (53) f_1 satisfies the differential equation

$$f_1'' - 4x f_1' - \frac{\pi^4}{2} f_1^3 = 0, \quad (55)$$

subject to the boundary condition

$$f_1(x) \underset{x \rightarrow \infty}{\sim} -\sqrt{\frac{2}{\pi^5}} \frac{e^{-\frac{4}{3}x^{3/2}}}{x^{1/4}}, \quad (56)$$

(here we have corrected a factor of π^2 in Eq. (5.13) of [22] in writing (55), while in writing (56) we have changed the sign in Eq. (5.14); to see that the latter is needed compare Eq. (5.14) with Eq. (5.4)).

As noted in [22], (55) can be identified with the Painlevé II equation (7) in the case $\alpha = 0$. Explicitly, the latter is obtained upon the substitutions

$$x = 2^{-2/3} t, \quad f_1(x) = -\frac{2^{5/3}}{\pi^2} u(t) \quad (57)$$

and furthermore, upon recalling the asymptotic expansion

$$\text{Ai}(t) \underset{t \rightarrow \infty}{\sim} \frac{e^{-2t^{3/2}/3}}{2\sqrt{\pi t^{1/4}}},$$

a boundary condition consistent with (9) is obtained.

Substituting (53) in (52) implies that with x as specified by (54) fixed [22, below (5.16)]

$$\frac{d^2}{dA^2} \log \mathcal{Z}(A; \mathbf{U}(N)) \sim \frac{N^2}{2\pi^4} \left(1 - \frac{2x}{N^{2/3}} - \frac{\pi^4}{2N^{2/3}} f_1^2(x) \right),$$

or equivalently, upon recalling (54) and making use of the variables (57),

$$\frac{d^2}{dt^2} \log e^{-A^2 N^2 / 4\pi^4} \mathcal{Z}(A; \mathbf{U}(N)) \Big|_{A=\pi^2 - t\pi^2 / (2N)^{2/3}} = -q^2(t), \quad (58)$$

where $q(t)$ is as in (6).

The relation (42) now tells us that for the ratio of return probabilities we have that for $N \rightarrow \infty$

$$\frac{d^2}{dt^2} \log \tilde{G}_N(2\sqrt{N}(1 + t/2(2N)^{2/3})) = -q^2(t). \quad (59)$$

But the distribution (11) for the scaled largest eigenvalue in the GUE satisfies this same relation, and so we have

$$\lim_{N \rightarrow \infty} \tilde{G}_N(2\sqrt{N}(1 + t/2(2N)^{2/3})) = \mathcal{F}_2(t). \quad (60)$$

Here use has also been made of the fact, which follows from (45), that the left-hand side (lhs) tends to 1 as $t \rightarrow \infty$.

We know from previous studies of large deviation formulas associated with the largest eigenvalue of a random matrix [12, 13, 52, 35, 40, 7], where the transition region is specified by the Tracy-Widom scaling function, that the expansion of the large deviation functions around the transition point coincides with the tail behaviours of the transition region Tracy-Widom scaling function. Interestingly, this property holds in the present setting for one side (left) of the tails only. Thus making use of the expansions (38) and (48) we obtain

$$\tilde{G}_N(2\sqrt{N}(1 + t/2(2N)^{2/3})) \underset{t \rightarrow -\infty}{\sim} e^{\frac{t^3}{12}} \quad (61)$$

$$\tilde{G}'_N(2\sqrt{N}(1 + t/2(2N)^{2/3})) \underset{t \rightarrow \infty}{\sim} e^{-\frac{2t^{3/2}}{3}} \quad (62)$$

with the first of these only the precise tail form $\mathcal{F}_2(t)$ [49]. The $t \rightarrow \infty$ tail of $\mathcal{F}_2(t)$ has the leading form $e^{-\frac{4t^{3/2}}{3}}$, and so differs by a factor of 2 in the exponent. This is the indication of an interesting crossover which we describe here qualitatively [55]. Close to $L = 2\sqrt{N}$, with $L - 2\sqrt{N} \sim \mathcal{O}(N^{-1/6})$, a calculation of $\tilde{G}_N(2\sqrt{N}(1 + t/2(2N)^{2/3}))$ beyond leading order shows that

$$\log \tilde{G}_N(2\sqrt{N}(1 + t/2(2N)^{2/3})) = \log(\mathcal{F}_2(t)) + N^{-1/3}(-1)^{N+1}g(t) + o(N^{-1/3}), \quad (63)$$

where the function $g(t)$ can be expressed explicitly in terms of $q(t)$ and behaves asymptotically as [55]

$$g(t) \sim e^{-\frac{2}{3}t^{3/2}}, \text{ when } t \rightarrow \infty. \quad (64)$$

What happens when one increases L from the critical region $L - 2\sqrt{N} \sim \mathcal{O}(N^{-1/6})$ towards the large deviation regime in the right tail, $L > 2\sqrt{N}$ (see Fig. 4)? As L increases away from $L_c(N) = 2\sqrt{N}$, the amplitude of the second term in the rhs of Eq. (63), which is oscillating with N , increases relatively to the amplitude of the first term. And at some value $L \equiv L_{\text{cross}}(N)$, it becomes larger than the first one: in the large deviation regime it becomes the leading term (still oscillating with N), as given in Eq. (49). On the other hand, the first term in the rhs of Eq. (63), $\log(\mathcal{F}_2(t))$, is subdominant for $L > L_{\text{cross}}(N)$ and becomes actually the term of order $\mathcal{O}(e^{-\frac{4\pi^2 N}{A}\gamma(\frac{A}{\pi^2})})$ in Eq. (47). Hence, there is a crossover between the two terms in the rhs of Eq. (63) as L crosses the value $L_{\text{cross}}(N)$. Balancing these two terms and making use of the leading behavior of the right tail of $\mathcal{F}_2(t)$ together with the asymptotic behavior of $g(t)$ given in Eq. (64), one obtains an estimate of $L_{\text{cross}}(N)$ as [55]

$$L_{\text{cross}}(N) - 2\sqrt{N} \sim N^{-1/6} (\log N)^{2/3}. \quad (65)$$

Note that such a crossover is absent in the distribution of the largest eigenvalue of GUE random matrices and it is thus a specific feature of this vicious walkers problem.

4 Brownian walkers with absorbing walls: Model II

In Model II, we have non-intersecting Brownian motions on $[0, L]$ with absorbing boundary conditions. Comparing (17) and (30) we have the exact identity

$$\tilde{F}_N(L) = \frac{B_N}{\hat{c}_N} e^{\pi^2 N(N+1/2)(N+1)/6L^2} L^{-2N^2-N} \mathcal{Z}\left(\frac{2\pi^2 N}{L^2}; \text{Sp}(2N)\right), \quad (66)$$

where the constants B_N and \hat{c}_N are independent of L . As in the $U(N)$ case, thanks to this identity and known large N properties of the partition function $\mathcal{Z}(A; \text{Sp}(2N))$ provides us new large N results for normalized reunion probability $\tilde{F}_N(L)$ as a function L . In addition, in this case, $\tilde{F}_N(L)$ is also identical to the distribution of the maximal height H_N for non-intersecting Brownian excursions. We thus get, as a bonus, new asymptotic results for the height distribution: both in the critical regime where its scaling behaviour is described by the Tracy-Widom

distribution $\mathcal{F}_1(t)$ of GOE matrices, as well as in its large deviation tails. As in the previous section, below we first discuss the large deviation tails and then the behavior in the critical regime.

4.1 Large deviation tails

In this case, using the correspondence $A = 2\pi^2 N/L^2$, the critical point $A = \pi^2$ in the gauge theory corresponds to a critical value $L = \sqrt{2N}$. For convenience, we then scale $L = \sqrt{2N}h$ so that the critical point is now at $h = 1$. Choosing $\hat{c}_N = 1$, (66) then reads

$$\tilde{F}_N(\sqrt{2N}h) = \frac{B_N}{(\sqrt{2N}h)^{2N^2+N}} e^{-\pi^2(N+1/2)(N+1)/12h^2} \mathcal{Z}\left(\frac{\pi^2}{h^2}; \text{Sp}(2N)\right). \quad (67)$$

The partition function (30) can be analyzed in terms of the same orthogonal polynomials appearing in (52). Thus one has [11, eq. (23)]

$$\frac{d^2}{dA^2} \log \mathcal{Z}(A; \text{Sp}(2N)) = \frac{1}{4N^2} R_N^{(+)} R_{N+1}^{(+)}. \quad (68)$$

Let us first consider the behavior $h > 1$ (corresponding to the weak coupling phase $A < \pi^2$). Knowledge of the large N form of the polynomials $R_N^{(+)}$ for $A < \pi^2$ allows the derivation of the expansion [11, eq. (30)]

$$\begin{aligned} \log e^{-A(N+1/2)(N+1)/12} \mathcal{Z}(A; \text{Sp}(2N)) &= \left(-N^2 - \frac{N}{2}\right) \log A \\ &+ \frac{A}{8\pi^2\sqrt{\pi N}} \left(1 + \frac{1}{\sqrt{1 - A/\pi^2}}\right) \left(1 - \frac{A}{\pi^2}\right)^{-1/4} e^{-\frac{2\pi^2 N}{A}\gamma(A/\pi^2)} + \mathcal{O}\left(e^{-\frac{4\pi^2 N}{A}\gamma(A/\pi^2)}\right), \end{aligned} \quad (69)$$

up to terms independent of A (the latter depend on the precise form of \hat{c}_N in (30)), where $\gamma(x)$ is given by (48).

Substituting (69) in (67) we see the first term in the former cancels, allowing us to conclude that for $h > 1$

$$\tilde{F}_N(\sqrt{2N}h) \underset{N \rightarrow \infty}{\sim} 1 \quad (70)$$

and furthermore

$$\tilde{F}'_N(\sqrt{2N}h) \underset{N \rightarrow \infty}{\sim} e^{-2Nh^2\gamma(1/h^2)}, \quad (71)$$

where $\gamma(x)$ is given in (48). Note that in Model II, the derivative $\tilde{F}'_N(L)$ has the interpretation of the probability density of the maximal height H_N since $\tilde{F}_N(L) = \Pr(H_N \leq L)$ is the cumulative distribution of height H_N . It is useful now to

compare (71) and the analogous result (49) in Model I. Note that, unlike in (49), the derivative does not oscillate in sign as N varies. This is reassuring since in the present case, $\tilde{F}'_N(L)$ has the meaning of a probability density which is necessarily positive.

We turn now to the large deviation formula for $h < 1$. To derive this large deviation formula in this case, in principle we need to repeat, for $Sp(2N)$, the strong coupling calculation of Douglas and Kazakov [15] originally done for the $U(N)$ case. In practice, however, as we show below, one can avoid repeating this calculation by noting an analogy in the Coloumb gas representation in the two cases and thereby relating the behavior of $\mathcal{Z}(A; Sp(2N))$ and $\mathcal{Z}(A; U(N))$ in the strong coupling regime. As a result, we can then directly use the results of Douglas and Kazakov stated in (36).

Let us first recall that for the ratio of reunion probabilities for Brownian walkers on the circle, we relied on the knowledge of the leading asymptotic form of the Yang-Mills partition function $\mathcal{Z}(A; U(N))$ for $A > \pi^2$ known from [15]. The latter in turn is deduced using the constrained continuum saddle point formula in the Coloumb gas representation

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N^2} \log e^{A(N^2-1)/24} \mathcal{Z}(A; U(N)) \\ &= -\frac{A}{2} \int_{-c}^c x^2 \rho(x) dx + \int_{-c}^c dx \int_{-c}^c dy \rho(x) \rho(y) \log |x - y|, \end{aligned} \quad (72)$$

where the charge density $\rho(x)$ maximizes the rhs subject to the constraints

$$\int_{-c}^c \rho(x) dx = 1, \quad 0 \leq \rho(x) \leq 1, \quad (73)$$

the latter being a direct signature of the spacing in the lattice gas being $1/N$ (recall the discussion above (36)).

In the case of (30), we begin by supposing all $n_i > 0$, which simply alters \hat{c}_N . Then, with $n_i/(2N)$ regarded as the continuous variable, the analogue of (72) reads

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N^2} \log e^{-A(N+1/2)(N+1)/12} \mathcal{Z}(A; Sp(2N)) \\ &= -A \int_0^d x^2 \tilde{\rho}(x) dx + \int_0^d dx \int_0^d dy \tilde{\rho}(x) \tilde{\rho}(y) \log |x^2 - y^2|, \end{aligned} \quad (74)$$

where the density $\tilde{\rho}(x)$ maximizes the rhs subject to the constraints

$$\int_0^d \tilde{\rho}(x) dx = 1, \quad 0 \leq \tilde{\rho}(x) \leq \frac{1}{2}. \quad (75)$$

If we take $\tilde{\rho}(x) = \tilde{\rho}(-x)$, $\rho(x) = 2\tilde{\rho}(x)$ and $d = c$ then the rhs of (74) can be rewritten

$$-A \int_{-c}^c x^2 \tilde{\rho}(x) dx + 2 \int_{-c}^c dx \int_{-c}^c dy \tilde{\rho}(x) \tilde{\rho}(y) \log|x-y|, \quad (76)$$

subject to (73). Since the symmetry of the problem implies $\rho(x)$ is even in (72), this is just twice the rhs of (72), so we conclude

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N^2} \log e^{-A(N+1/2)(N+1)/12} \mathcal{Z}(A; \text{Sp}(2N)) \\ = 2 \lim_{N \rightarrow \infty} \frac{1}{N^2} \log e^{A(N^2-1)/24} \mathcal{Z}(A; \text{U}(N)). \end{aligned} \quad (77)$$

Using a similar argument an inter-relation of this type was first noted in [10], although there it is claimed in Eq. (3.7) that A must be replaced by $A/2$ on the rhs; this in turn is contradicted by the later paper of the same authors [11]. We remark that for $A < \pi^2$ (77) is consistent with (69) and the appropriate case of (36).

For $h < 1$, it follows from (77), (36) in the case $A > \pi^2$, and (67) that

$$\tilde{F}_N(\sqrt{2N}h) \sim e^{-2N^2(F_-(1/h^2) - F_+(1/h^2))}. \quad (78)$$

4.2 Double scaling limit

Having obtained the large N asymptotic behavior of $\tilde{F}_N(L)$ as a function of L in the left ($L < \sqrt{2N}$) and the right ($L > \sqrt{2N}$) tails, let us now focus on the behavior of $\tilde{F}_N(L)$ in the vicinity of the critical point, i.e., when L is close to $\sqrt{2N}$. As in the $\text{U}(N)$ case, this corresponds to the double scaling regime near the critical point $A = \pi^2$ in the $\text{Sp}(2N)$ gauge theory. Using results from the gauge theory, we will show here in a narrow region $|L - \sqrt{2N}| \sim N^{-1/6}$ around the critical point $L = \sqrt{2N}$, $\tilde{F}_N(L)$ has the following scaling behavior

$$\tilde{F}_N(L) \rightarrow \mathcal{F}_1 \left(2^{11/6} N^{1/6} |L - \sqrt{2N}| \right), \quad (79)$$

where the scaling function $\mathcal{F}_1(t)$ is precisely the Tracy-Widom distribution function for GOE random matrices defined in (12).

Using results from [15], the double scaling limit of (68) has been analyzed in [11]. In particular, with

$$x_{2N} := n_c^{2/3} \left(1 - \frac{2N}{n_c} \right), \quad n_c := \frac{2N\pi^2}{A}, \quad (80)$$

one has from Eq. (34) of [11] that in the double scaling limit $N \rightarrow \infty$, x_{2N} fixed

$$\frac{d^2}{dA^2} \log \mathcal{Z}(A; \text{Sp}(2N)) = \frac{n_c^4}{16\pi^4 N^2} \left(1 - \frac{2x_{2N}}{n_c^{2/3}} - \frac{\pi^4}{2n_c^{3/2}} f_1^2(x_{2N}) - \frac{\pi^2}{n_c^{2/3}} f_1'(x_{2N}) \right). \quad (81)$$

To make use of (81), we first note that with the definition

$$W_N(A) := \log e^{-A(N+1/2)(N+1)/12} \mathcal{Z}(A; \text{Sp}(2N)), \quad (82)$$

we have from (17)

$$\frac{\partial^2}{\partial L^2} \log \tilde{F}_N(L) = \frac{2N^2 + N}{L^2} + \frac{12N\pi^2}{L^4} W_N' \left(\frac{2N\pi^2}{M^2} \right) + \frac{16N^2\pi^4}{L^6} W_N'' \left(\frac{2N\pi^2}{L^2} \right). \quad (83)$$

Suppose now we fix

$$L^{4/3} \left(1 - \frac{2N}{L^2} \right) := x_{2N}. \quad (84)$$

For large N , upon writing the second factor as the difference of two squares and setting $L = \sqrt{2N}$ where this does not lead to a zero term, this is equivalent to setting

$$x_{2N} = 2^{7/6} N^{1/6} (L - \sqrt{2N}). \quad (85)$$

Recalling (67), and with $n_c = (\sqrt{2N}h)^2$ in (80), it follows from (81), (83) and (84) that

$$\frac{\partial^2}{\partial L^2} \log \tilde{F}_N(L) = -\frac{4N^2 + 3N}{L^2} + L^2 \left(1 - 2\frac{x_{2N}}{L^{4/3}} \right) - L^{2/3} \left(\frac{\pi^4}{2} f_1^2(x_{2N}) + \pi^2 f_1'(x_{2N}) \right). \quad (86)$$

But use of (85) shows that

$$-\frac{4N^2 + 3N}{L^2} + L^2 \left(1 - 2\frac{x_{2N}}{L^{4/3}} \right) = \mathcal{O}(N^{1/6}), \quad (87)$$

allowing these terms to be ignored in (86), and leaving us with

$$\frac{\partial^2}{\partial L^2} \log \tilde{F}_N(L) = -L^{2/3} \left(\frac{\pi^4}{2} f_1^2(x_{2N}) + \pi^2 f_1'(x_{2N}) \right). \quad (88)$$

Finally, use (85) to replace the derivative with respect to L on the lhs of (88) by a derivative with respect to x_{2N} , and substitute for $x_{2N} = x$ according to (57), and $f_1(x)$ for $u(t) = q(t)$ also according to (57). We then see that (88) implies

$$\frac{d^2}{dt^2} \log \tilde{F}_N \left(\sqrt{2N} (1 + t/(2^{7/3} N^{2/3})) \right) = -\frac{1}{2} \left(q^2(t) + q'(t) \right). \quad (89)$$

Comparison with (11) shows that the distribution of the scaled largest eigenvalue in the GOE satisfies the same relation, and so we have

$$\tilde{F}_N\left(\sqrt{2N}(1+t/(2^{7/3}N^{2/3}))\right) = \mathcal{F}_1(t). \quad (90)$$

As in deducing (60) from (59), to deduce (90) from (89) we have used also the fact that the lhs must tend to 1 as $t \rightarrow \infty$, which is a consequence of (70).

We saw in the case of the ratio of return probabilities for the walkers on the circle that the large deviation formula relating the values smaller than the mean is connected to the left tail of the double scaling distribution about the mean. The present problem of the cumulative distribution for the maximum displacement of non-intersecting Brownian walkers near a wall exhibits the same feature. Thus making use of the expansion (38) in (78) shows

$$\tilde{F}_N\left(\sqrt{2N}(1+t/(2^{7/3}N^{2/3}))\right) \underset{t \rightarrow -\infty}{\sim} e^{\frac{t^3}{24}}, \quad (91)$$

which is indeed the leading order tail form of $\mathcal{F}_1(t)$ [50, 44]. On the other hand, the right tail of the distribution $\tilde{F}_N(L)$ for $L > \sqrt{2N}$ exhibits a crossover exactly similar to the one described above for non-intersecting Brownian motions on a circle (63, 65) [55]. Note however that in that case, the second term as in Eq. (63) does not oscillate with N because in that case $\tilde{F}_N(L)$ has an interpretation in terms of cumulative distribution.

The result (90) is in keeping with known results about fluctuating interfaces belonging to the universality class of the Kardar-Parisi-Zhang (KPZ) equation in $1+1$ dimensions. Indeed, the top path of such watermelons configuration (see Fig. 2) can be mapped, in the limit $N \rightarrow \infty$, onto the height field of such KPZ interface in the so-called "droplet" (*i.e.* curved) geometry [41]. The extreme value statistics of such interface in the KPZ universality class and in curved geometry has recently attracted some attention [28, 43]. In particular, using the fact that the maximal value of the height field in the droplet geometry can be mapped onto the height field (at a given point) in the flat geometry [33] it was shown, albeit indirectly in Ref. [28], that the distribution of H_N (see Fig. 2), correctly shifted and scaled, is indeed described by $\mathcal{F}_1(t)$. Here, we obtain this result by a direct computation of the distribution of H_N in the large N limit. Moreover the $\mathcal{F}_1(t)$ fluctuations have previously been established in the case of the distribution of the displacement M of the right-most walker amongst N returning vicious walkers in discrete time and in the presence of a wall, and with the technical requirement that twice the number of walkers be greater than the total number of steps [4]. This model gives rise the matrix integral

$$\left\langle \text{Tr} S^{2N} \right\rangle_{S \in \text{Sp}(2M)}, \quad (92)$$

which upon Poissonization in N is the symplectic analogue of (1). In fact the Poissonized form of (92) appears in Hammersley model of directed last passage percolation as revised in Section 1.1, but with the points confined to be below the line $y = x$ in the square [5], [19, §10.7.1].

5 Conclusion

To conclude, we have studied the normalized reunion probability of N non-intersecting Brownian motions confined on a line segment $[0, L]$ with three different types of boundary conditions : (I) periodic (where the Brownian walkers are thus moving on a circle of radius $L/2\pi$) (13), (II) absorbing boundary conditions at both extremities 0 and L (16) and (III) reflecting boundary conditions at both ends (18). We have shown that, in each of these models, this quantity is given (up to a prefactor that we have computed) by the partition function of $2-d$ Yang-Mills theory on the sphere with a given gauge group, and computed with the heat-kernel action. We have found that models I, II and III correspond respectively to the group $U(N)$ (32), $Sp(2N)$ (33) and $SO(2N)$ (34). Borrowing results from these different field theories, we have shown that, in the large N limit, these reunion probabilities exhibit a third-order phase transition as L crosses a critical value $L_c \sim \sqrt{N}$. The region corresponding to $L > L_c$, which corresponds to the weak coupling regime in the Yang-Mills theory, describes the right tail of this normalized reunion probability, while the region $L < L_c$, which corresponds to the strong coupling regime, describes its left tail (Fig. 4). In the critical region of width $N^{-1/6}$, close to L_c , one finds that the reunion probability, correctly shifted and scaled, converges to the Tracy-Widom distribution corresponding respectively to GUE, $\mathcal{F}_2(t)$, in model I and to GOE, $\mathcal{F}_1(t)$, in model II and III. One of the main achievements of this paper to relate the Painlevé equation which describes the singularity of the free energy in the double scaling limit of these $2-d$ Yang-Mills theories with the one defining the Tracy-Widom distributions. In the case of model II, the normalized reunion probability has the interpretation of the cumulative distribution of the maximal height of the corresponding watermelon configuration (Fig. 2). Our results thus show directly that this cumulative distribution is given in the large N limit by $\mathcal{F}_1(t)$, a result was obtained before in a rather indirect way in Ref. [28].

In this paper, we have thus presented the correspondence between boundary conditions (in the vicious walkers models) and gauge groups (in Yang-Mills theories in two dimensions on a sphere) and discussed its consequences but the deep reason behind it deserves certainly further study. An alternative way to explore these connections could be to study the relations between vicious walkers models and Chern-Simons theory as in Ref. [25, 26, 48]. Yet another point of view could be

to adopt the formulation of these vicious walkers problems in terms of Dyson's Brownian motion. Indeed, it can be shown that the propagator of this process, expressed as a path integral, is precisely given by the partition function of Yang-Mills theory on the sphere with the appropriate gauge group, depending on the boundary conditions [38]. We hope that this will stimulate further works along these directions.

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A Ratio of reunion probabilities for non-intersecting Brownian motions on a circle

In this appendix, we derive the formula given in Eq. (14) for the ratio $\tilde{G}_N(L) = \tilde{R}_L^I(1)/\tilde{R}_\infty^I(1)$ introduced in Eq. (13). We consider N non-intersecting Brownian motions on a line segment $[0, L]$ with periodic boundary conditions or equivalently on a circle of radius $L/2\pi$, starting in the vicinity of the origin at time $\tau = 0$. The reunion probability $\tilde{R}_L^I(1)$ denotes the probability that the walkers return to their initial configuration at time $\tau = 1$.

Let us begin with the case of N free Brownian motions, with a diffusion constant $D = 1/2$, on a circle of radius $L/2\pi$. The position of the N walkers are thus labelled by the angles $\theta_1, \dots, \theta_N$. We denote by $P_N(\theta_1, \dots, \theta_N; t | \rho_1, \dots, \rho_N; 0)$ the probability that the positions of the N walkers are $\theta_1, \dots, \theta_N$ at time t , given that their positions were ρ_1, \dots, ρ_N at initial time. It is easy to see that $P_N(\theta_1, \dots, \theta_N; t | \rho_1, \dots, \rho_N; 0)$ satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial t} P_N = \frac{2\pi^2}{L^2} \sum_{k=1}^N \frac{\partial^2}{\partial \theta_k^2} P_N,$$

$$P_N(\theta_1, \dots, \theta_N; t = 0 | \rho_1, \dots, \rho_N; 0) = \prod_{k=1}^N \delta(\theta_k - \rho_k), \quad (93)$$

which simply comes from the expression of the bi-dimensional Laplacian in terms of polar variables (we recall that the radius of the circle is $L/2\pi$ and the diffusion

coefficient is $D = 1/2$), together with the constraint of periodicity

$$P_N(\cdots, \theta_k + 2\pi, \cdots; t | \rho_1, \cdots, \rho_N; 0) = P_N(\cdots, \theta_k + 2\pi, \cdots; t | \rho_1, \cdots, \rho_N; 0), \quad \forall 1 \leq k \leq N, \quad (94)$$

and similarly for a shift of 2π of the variables ρ_k 's. Therefore, from Eq. (93) and Eq. (94), P_N can be written as the propagator, in imaginary time, of N independent quantum free particles on a circle of circumference L . Using path integral techniques, one thus writes (using the notation $\boldsymbol{\theta} \equiv (\theta_1, \cdots, \theta_N)$)

$$P_N(\theta_1, \cdots, \theta_N; t | \rho_1, \cdots, \rho_N; 0) = \langle \boldsymbol{\theta} | e^{-t\hat{H}_L} | \boldsymbol{\rho} \rangle, \quad (95)$$

with $\hat{H}_L = \sum_{i=1}^N \hat{h}_{L,i}$ where $\hat{h}_{L,i} = -\frac{2\pi^2}{L^2} \frac{\partial^2}{\partial \theta_i^2}$ has to be understood as the Hamiltonian of a free particle on a circle of circumference L so that the allowed eigenvalues of $\hat{h}_{L,i}$ are $E_{n_i} = \frac{2\pi^2 n_i^2}{L^2}$ associated to the eigenvectors $\phi_{n_i}(\theta) = \frac{1}{\sqrt{2\pi}} e^{in_i \theta}$, $n_i \in \mathbb{N}$. Therefore one has in that case

$$P_N(\theta_1, \cdots, \theta_N; t | \rho_1, \cdots, \rho_N; 0) = \sum_E \Psi_E(\theta_1, \cdots, \theta_N) \Psi_E^*(\rho_1, \cdots, \rho_N) e^{-Et}, \quad (96)$$

with $E = E_{n_1} + \cdots + E_{n_N}$ and $\Psi_E(\theta_1, \cdots, \theta_N) = \langle \boldsymbol{\theta} | E \rangle$ is the manybody eigenfunction of \hat{H}_L . For independent Brownian motions (without the non-crossing condition), $\Psi_E(\theta_1, \cdots, \theta_N)$ is simply the product of the single particle wave function $\Psi_E(\theta_1, \cdots, \theta_N) = \prod_{i=1}^N \phi_{n_i}(\theta_i)$.

We can now consider the problem of N non-intersecting Brownian motions on a circle of circumference L and study the corresponding propagator $P_N(\boldsymbol{\theta}; t | \boldsymbol{\rho}; 0)$. It satisfies the same equations as before (93, 94) together with the additional non-crossing constraint:

$$P_N(\theta_1, \cdots, \theta_N; t | \rho_1, \cdots, \rho_N; 0) = 0 \text{ if } \theta_j = \theta_k \text{ for any pair } j, k. \quad (97)$$

Following Ref. [47, 40], this propagator can be computed using the path-integral formalism explained above (96) where, to incorporate the non-colliding condition, the many-body wave function $\Psi_E(\theta_1, \cdots, \theta_N)$ must be Fermionic, *i.e.* it vanishes if any of the two coordinates are equal. This anti-symmetric wave function is thus constructed from the one particle wave functions ϕ_{n_i} of \hat{h}_i by forming the associated Slater determinant. Therefore one has, in that case

$$\psi_E(\boldsymbol{\theta}) = \frac{1}{\sqrt{N!}} \det_{1 \leq j, k \leq N} \phi_{n_j}(\theta_k), \quad E = \frac{2\pi^2}{L^2} \sum_{k=1}^N n_k^2. \quad (98)$$

From the propagator P_N , we can now compute the reunion probability, which is the probability that all the N walkers start and end at the same position on

the circle, say $\theta_1 = \theta_2 = \dots = \theta_N = 0$, on the unit time interval. However, such a probability is ill defined for a system in continuous space and time. We can go around this problem by assuming that the starting and finishing positions (angles) of the N walkers are $0 < \epsilon_1 < \epsilon_2 < \dots < \epsilon_N$ and only at the end take the limit $\epsilon_i \rightarrow 0$. Therefore we can compute the ratio of reunion probabilities $\tilde{G}_N(L)$ as

$$\tilde{G}_N(L) = \lim_{\epsilon_i \rightarrow 0} \frac{\langle \epsilon | e^{-\hat{H}_L} | \epsilon \rangle}{\langle \epsilon | e^{-\hat{H}_\infty} | \epsilon \rangle}, \quad (99)$$

where \hat{H}_∞ denotes the Hamiltonian of the N walkers on the full real axis. Using the expressions in Eqs (96, 98), one checks that, in the limit $\epsilon_i \rightarrow 0$, powers of ϵ_i 's cancel between the numerator and the denominator in Eq. (99) yielding the expression for $\tilde{G}_N(L)$ given in Eq. (14) in the text.

For instance, for $N = 1$ one has

$$\tilde{G}_1(L) = \frac{\sqrt{2\pi}}{L} \sum_{n=-\infty}^{\infty} e^{-\frac{2\pi^2}{L^2} n^2}, \quad (100)$$

which can also be written, using the Poisson summation formula

$$\tilde{G}_1(L) = \sum_{n=-\infty}^{\infty} e^{-\frac{L^2}{2} n^2}. \quad (101)$$

In particular, we obtain the large L behavior as

$$\tilde{G}_1(L) = 1 + 2e^{-\frac{L^2}{2}} + \mathcal{O}(e^{-2L^2}), \quad (102)$$

showing explicitly that $\tilde{G}_1(L)$ does not have the meaning of a cumulative distribution.

B Reunion probability with absorbing and reflecting boundary conditions

Here we briefly outline the derivations of the results for $\tilde{F}_N(L)$ in (17) and $\tilde{E}_N(L)$ in (19). In the first case, we again have N non-intersecting Brownian motions on the line segment $[0, L]$ with absorbing boundary conditions at 0 and L . The walkers all start at time $\tau = 0$ in the vicinity of the origin and we want to compute their reunion probability $R_L^{II}(1)$ near the origin after time $\tau = 1$. The calculation proceeds in the same way as in the periodic case in the previous appendix. One writes the Fokker-Planck equation for the probability density

$P_N(x_1, x_2, \dots, x_N; t | y_1, \dots, y_N; 0)$ of reaching $\{x_1, x_2, \dots, x_N\}$ at time t starting from the initial positions $\{y_1, y_2, \dots, y_N\}$. This reads

$$\frac{\partial}{\partial t} P_N = \frac{1}{2} \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2} P_N ,$$

$$P_N(x_1, \dots, x_N; t = 0 | y_1, \dots, y_N; 0) = \prod_{k=1}^N \delta(x_k - y_k). \quad (103)$$

One then writes the solution using path integrals exactly as in (96). The rest of the calculation is similar as in Appendix A. The only difference is that in constructing the Slater determinant, one now has to use the normalized single particle wave function as

$$\phi_{n_k}(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n_k \pi x}{L}\right) , \quad (104)$$

which satisfies the absorbing boundary condition $\phi_{n_k}(x) = 0$ at $x = 0$ and $x = L$. Using this, one just repeats the calculation of Appendix A to derive the result for $\tilde{F}_N(L)$ in (17).

In the reflecting case, the only change is again in the normalized single particle wave function that reads

$$\phi_{n_k}(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{n_k \pi x}{L}\right) \quad (105)$$

and satisfies the reflecting boundary condition $\partial_x \phi_{n_k}(x) = 0$ at $x = 0$ and $x = L$. Repeating the rest of the calculation with this modification as in Appendix A, one easily derives the result for $\tilde{E}_N(L)$ in (19).

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