On the distribution of the Wigner time delay in one-dimensional disordered systems

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Abstract

We consider the scattering by a one-dimensional random potential and derive the probability distribution of the corresponding Wigner time delay. It is shown that the limiting distribution is the same for two different models and coincides with the one predicted by random matrix theory. It is also shown that the corresponding stochastic process is given by an exponential functional of the potential.

IPNO/TH 97-15

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1 Introduction

The concept of time delay introduced long ago by Eisenbud and Wigner has recently received renewed attention. In a scattering process the time delay $\tau(k)$ is related to the time spent in the interaction region by a wavepacket[4] with energy peaked at $E = k^2$. It can be expressed in terms of the derivative of the S matrix with respect to the energy. If the system is described by a finite number N of channels, the N time delays are the eigenvalues of the matrix $-iS^\dagger \frac{\partial S}{\partial E}$. In the context of chaotic scattering a statistical approach based on random matrix theory allows a determination of the complete distribution of time delays (for a review see [3]). This problem was first studied in [4] by a supersymmetric approach, and in [5] by using a statistical analysis. This latter work provided a physical derivation of the one-channel case for $\beta = 2$ and gave the distribution for other universality classes. It then recently served as a starting point for [6] where the N-channel distribution is shown to be given by the Laguerre ensemble of random matrix theory. A slightly different approach has been developed in the context of quasi one-dimensional mesoscopic systems. Quite recently a resonant transmission model was proposed to account for the different behaviours observed in the metallic and insulating regimes [7].

In this paper we consider a one-dimensional model with a random potential whose support is a finite segment of length L. A hard wall condition at the origin reduces the problem to a scattering problem on the half line. Using standard methods we write down a set of two coupled stochastic differential equations satisfied by the phase and its derivative with respect to the energy which is related to the time delay. In the high energy (or weak disorder) limit we may take advantage of the fact that the phase is a rapid variable with a uniform distribution. One finally ends up with a stochastic differential equation for the time delay which coincides with the one derived in [8, 9] and which can be integrated explicitly in the case where the potential is white noise. The resulting expression is given in terms of an exponential functional of a Brownian motion. An analysis of the same model with a different type of disorder yields the same functional and consequently the same stationary distribution. Our work therefore brings some support to the conjecture that these distributions are universal. Another aim of this work is to draw attention to the fact that the same probability distribution also arises in other physical problems, for instance in the context of diffusion in a random medium.

2 The model with a white noise potential

Consider the Schrödinger equation on the half line $x \geq 0$

$$-\frac{d^2}{dx^2}u(x) + V(x)u(x) = k^2 u(x)\,. \tag{1}$$

We assume that the potential has its support on the interval $[0, L]$ and impose Dirichlet boundary condition at the origin $u(0) = 0$. Therefore for $x > L$ scattering states of the form $u(x) = \mathrm{e}^{-ikx} + \mathrm{e}^{ikx+i\theta(k)}$ represent the superposition of an incoming plane wave and a reflected plane wave. In this case the reflection coefficient $r(k) = \mathrm{e}^{i\theta(k)}$ is of unit modulus since there is only backward scattering. All information on the scattering process

[4] For a detailed and critical analysis of this concept see [1] and [2].

is contained in the phase shift $\theta(k)$. In particular the Wigner time delay is given by

$$\tau(k) = \frac{1}{2k}\frac{d\theta(k)}{dk} \,. \tag{2}$$

Our aim is to find the probability distribution of $\tau(k)$ in the case where the potential is a Gaussian white noise such that

$$\begin{aligned} \langle V(x)\rangle &= 0 \\ \langle V(x)\,V(y)\rangle &= \sigma_{\rm g}\,\delta(x-y)\,. \end{aligned} \tag{3}$$

A widely used method to deal with such problems is the invariant embedding approach [10, 9]. Here we will use a more straightforward derivation following [11].

The Ricatti variable $z(x) = \frac{d}{dx}\ln|u(x)|$ satisfies the first order nonlinear differential equation

$$\frac{dz(x)}{dx} = V(x) - k^2 - z(x)^2\,. \tag{4}$$

In the region where $V(x)$ vanishes one has

$$z(x) = \frac{-ik + ik\,{\rm e}^{2ikx+i\theta(k)}}{1+{\rm e}^{2ikx+i\theta(k)}}\,. \tag{5}$$

Inverting this relation we may express the reflection coefficient in terms of the Ricatti variable

$${\rm e}^{2ikx+i\theta(k)} = \frac{ik+z(x)}{ik-z(x)}\,. \tag{6}$$

Although this relation only holds in the region $x > L$, we may take it as a definition of the variable $\psi(x)$

$${\rm e}^{i\psi(x)} = \frac{ik+z(x)}{ik-z(x)} \tag{7}$$

in the region $0 \le x \le L$. By using equation (4) one gets the differential equation satisfied by $\psi(x)$

$$\frac{d\psi(x)}{dx} = 2k - \frac{1}{k}\Big(1+\cos\psi(x)\Big)V(x)\,. \tag{8}$$

Since the Ricatti variable is continuous even if the potential has a discontinuity this implies that the phase shift $\theta(k)$ is given by $\psi(L,k) - 2kL$. The reflection condition at the origin gives the boundary condition $\psi(0) = \pi$.

Let us introduce the derivative of ψ with respect to k: $Z(x,k) \equiv \frac{d}{dk}\psi(x,k)$ in terms of which the time delay reads

$$\tau(k) = \frac{Z(L,k) - 2L}{2k}\,. \tag{9}$$

The differential equation for Z now reads

$$\frac{dZ(x)}{dx} = 2 + \left[\frac{1}{k^2}\Big(1+\cos\psi(x)\Big) + \frac{1}{k}Z(x)\sin\psi(x)\right]V(x)\,. \tag{10}$$

From the boundary condition at $x = 0$ we may set $Z(0) = 0$.

If $V(x)$ is a Gaussian white noise equations (8,10) are coupled stochastic differential equations (in the Stratonovich sense) which give the phase and the time delay.

One then may pass from these two stochastic differential equations to a Fokker-Planck equation for the probability density of ψ and Z. In the high energy limit $k \gg \sigma_{\rm g}^{1/3}$ one can show that the variable ψ is a rapid variable uniformly distributed on the interval $[0, 2\pi]$. Moreover, since the rapid variable ψ and the slow variable Z decorrelate in this limit, then one may average over the rapid variable and eventually get the following Fokker-Planck equation for Z (a more detailed description of the procedure will be given in section 4)

$$\frac{\partial P(Z;x)}{\partial x} = \frac{\partial}{\partial Z}\left[\Big(\frac{\sigma_{\rm g}}{4k^2}Z - 2\Big)P(Z;x)\right] + \frac{\sigma_{\rm g}}{4k^2}\frac{\partial}{\partial Z}Z\frac{\partial}{\partial Z}Z\,P(Z;x)\,. \tag{11}$$

Up to an inessential term[5], this equation coincides with the one derived in [8] and [9]. It belongs to a more general class of Fokker-Planck equations that appear in the study of exponential functionals of Brownian motion with drift.

3 A representation of τ as an exponential functional of the Brownian motion

Equation (11) describes a special case of a class of stochastic processes which have been studied extensively in the mathematical [12] as well as in the physical literature. It can be cast into the equation studied by Schenzle et al. in the context of multiplicative stochastic processes [13]. More recently it was shown to arise in the context of diffusion in a random medium [14, 15]. The distribution of the flux of particles [16, 17] in a disordered sample of finite length or the waiting time distribution may be obtained by solving a generalization of equation (11). From our previous work [17, 18] one may write the general solution of (11) in the form

$$P(Z;L) = \frac{\lambda}{Z^2}{\rm e}^{-\frac{\lambda}{Z}} + \frac{2}{\pi Z}{\rm e}^{-\frac{\lambda}{2Z}}\int_0^\infty ds\,{\rm e}^{-\frac{L}{2\lambda}(1+s^2)}\frac{s}{1+s^2}\sinh\frac{\pi s}{2}\;\; W_{1,\frac{is}{2}}\left(\frac{\lambda}{Z}\right) \tag{12}$$

where $W_{\mu,\nu}(z)$ is a Whittaker's function and $\lambda = \frac{8k^2}{\sigma_{\rm g}}$ is the localization length at high energy ($k \gg \sigma_{\rm g}^{1/3}$). In the limit $L \to \infty$ the first term gives the limiting distribution which, as announced, coincides with the one channel distribution obtained in [4]. It is also in agreement with the result of Jayannavar et al. [8] in the high energy limit.

The general solution given above allows to study finite size effects. The first correction to the stationary distribution for Z may be calculated from (12)

$$P(Z;L) \underset{L\gg\lambda}{\simeq} \frac{\lambda}{Z^2}{\rm e}^{-\frac{\lambda}{Z}} + \sqrt{\frac{\pi}{2}}\left(\frac{\lambda}{L}\right)^{3/2}{\rm e}^{-\frac{L}{2\lambda}}\;W_{1,0}\left(\frac{\lambda}{Z}\right)\frac{{\rm e}^{-\frac{\lambda}{2Z}}}{Z} \tag{13}$$

One can also compute all the moments of the variable Z (or τ) [17, 18]. For n greater than one they all diverge exponentially as a function of the length of the sample.

$$\langle Z(L)\rangle = 2L \tag{14}$$

$$\langle Z(L)^n\rangle \simeq \frac{(n-2)!}{(2n-2)!}\lambda^n{\rm e}^{2n(n-1)\frac{L}{\lambda}} \tag{15}$$

[5] we do not write this term because it is of the same order as terms we neglected in the approximation of the decorrelation of ψ and Z.

It is interesting to remark that the limiting distribution is not of the log-normal form as could have been expected [9] from a simple resummation of the most divergent part of the moments.

Another by-product of our earlier work is to provide a representation of the process to which the random variable τ obeys. The corresponding stochastic differential equation associated with (11) may be written in the following form

$$\frac{dZ(x)}{dx} = 2 - \frac{\sigma_{\rm g}}{4k^2} Z(x) + \frac{1}{\sqrt{2}k} Z(x) V(x) \tag{16}$$

where $V(x)$ is the white noise (3). By integration one obtains Z and from (9) the time delay

$$\tau(k) = \frac{1}{k} \int_0^L dx \left({\rm e}^{\int_x^L dx' \left(\frac{V(x')}{\sqrt{2}\,k} - \frac{\sigma_{\rm g}}{4k^2} \right)} - 1 \right) . \tag{17}$$

A more satisfactory procedure to derive this formula is to start from the stochastic differential equations (8,10) and construct τ without using a Fokker-Planck equation. This approach was used by Faris and Tsay [19].

When writing this expression for $\tau(k)$ it should be kept in mind that although (17) is not true for every realization of $V(x)$, it nevertheless captures all the statistical properties of the process. In particular it allows the study of the various limiting cases. Since the potential $V(x)$ is a white noise, the first term in the exponential, which is a Brownian motion, is typically of order $\sqrt{\sigma_{\rm g} L/k^2}$. The argument of the exponential is roughly a function of the ratio of the localization length $\lambda(k) = 8k^2/\sigma_{\rm g}$ and the size of the system L. In particular if $\lambda \ll L$ the time delay is approximatively equal to $-L/k$ which is twice the length of the sample divided by the speed of the particle. This value for the time delay corresponds to a reflection at the sample edge. This agrees with the fact that if $\lambda \ll L$, one expects that the particle will only enter the sample for a negligeable length compared to L. In the other regime $\lambda \gg L$ expression (17) shows that the time delay is roughly zero. This is consistent with the fact that in this regime the sample is almost transparent to the particle.

Moreover if one considers the regime where $\lambda \ll L$, the stationary distribution for $Z(L)$ gives a typical value $Z_{\rm typ} \simeq 2\lambda$ corresponding to a typical time delay $\tau_{\rm typ} \simeq -\frac{L-\lambda}{k}$. Such a value for the time delay means that the particle enters in the disordered region for a typical length λ.

One may wonder to what extent the representation of Z as an exponential functional of the potential depends on the specific form of the noise that we have considered. Although we are not able to give a definite answer to this question we nevertheless notice that for any realization of $V(x)$ one may integrate equation (10) and get

$$Z(L) = 2 \int_0^L dx \left(1 + \frac{V(x)}{2k^2} (1 + \cos\psi(x)) \right) {\rm e}^{\frac{1}{k} \int_x^L dx'\, V(x') \sin\psi(x')} \tag{18}$$

hence

$$\tau(k) \underset{k \gg \sigma_{\rm g}^{1/3}}{\simeq} \frac{1}{k} \int_0^L dx \left({\rm e}^{\frac{1}{k} \int_x^L dx'\, V(x') \sin\psi(x')} - 1 \right) . \tag{19}$$

Up to a constant drift, at high energy, this is essentially of the form given in (17). The distribution and the asymptotic behaviour of exponential functionals of more general processes like Levy processes has been obtained in the mathematical literature [20, 21]. There it is shown that a Poisson process gives a limiting distribution whose tail still decays algebraically.

4 The supersymmetric model

4.1 The model

In this section we consider a different model for which one can also compute exactly the time delay distribution in the weak disorder limit. We show below that $\tau(k)$ is distributed with the same law as in the previous case.

We consider the one-dimensional Schrödinger Hamiltonian

$$H = -\frac{d^2}{dx^2} + \phi^2(x) + \phi'(x) \,. \tag{20}$$

As explained at length in [14, 22, 23, 24] this model arises in diverse areas of quantum mechanics ranging from the study of solitons in polymers to the study of classical diffusion in a random medium. Recently it was used in the context of one-dimensional spin chains [25] and isospectral periodic potentials [26].

In the following we consider the case where $\phi(x)$ is a Gaussian white noise with the moments

$$\begin{aligned} \langle \phi(x) \rangle &= 0 \\ \langle \phi(x)\, \phi(y) \rangle &= \sigma\, \delta(x-y) \,. \end{aligned} \tag{21}$$

The supersymmetric Hamiltonian (20) can be rewritten in the factorized form

$$H = Q^\dagger Q \tag{22}$$

where

$$Q = -\frac{d}{dx} + \phi(x) \tag{23}$$

$$Q^\dagger = \frac{d}{dx} + \phi(x) \,. \tag{24}$$

The density of states and the localization length for this model were first obtained in [27] and then rediscovered independently in [28]. In contrast with the previous model, in the high energy limit ($k \gg \sigma$) the localization length reaches the constant value $\lambda(k) = \frac{2}{\sigma}$.

The two other length scales of the problem are the size of the disorder region L and the de Broglie wavelength k^{-1}. In the following we choose to work in a high energy limit, i.e. when $k \gg \sigma$ and $k \gg L^{-1}$.

As in the previous section, the potential is non zero on the interval $[0, L]$ and there is a reflection condition at the origin.

4.2 Stochastic differential equations for the phase and the time delay

Using the factorization of the supersymmetric Hamiltonian, one may decouple the stationary Schrödinger equation $\left(-\frac{d^2}{dx^2}+\phi^2(x)+\phi'(x)\right)u(x)=k^2\,u(x)$ into two first order equations:

$$\frac{du(x)}{dx} = \phi(x)u(x) - kv(x) \tag{25}$$

$$\frac{dv(x)}{dx} = -\phi(x)v(x) + ku(x)\,. \tag{26}$$

If $\phi(x)$ has a discontinuity the two functions $u(x)$ and $v(x)$ are continuous. This suggests the introduction of the Ricatti variable $\zeta(x)=\frac{v(x)}{u(x)}$. It is straightforward to see that $\zeta(x)$ obeys the following first order non linear differential equation:

$$\frac{d\zeta(x)}{dx} = k - 2\phi(x)\zeta(x) + k\,\zeta(x)^2\,. \tag{27}$$

In the region where $\phi(x)$ is vanishing ($x>L$), the stationary scattering states can be expressed as

$$u(x) = \mathrm{e}^{-ikx} + \mathrm{e}^{ikx+i\theta(k)} \tag{28}$$

where $\theta(k)$ is the phase shift.

This leads to the change of variable:

$$\zeta(x) = i\frac{1-\mathrm{e}^{2ikx+i\alpha(x)}}{1+\mathrm{e}^{2ikx+i\alpha(x)}} \tag{29}$$

with the phase shift $\theta(k)$ given by $\alpha(L,k)$. Instead of $\alpha(x)$ it is in fact more convenient to introduce the variable $\psi(x)=\alpha(x)+2kx$ in terms of which the Ricatti variable is $\zeta(x)=\tan(\psi(x)/2)$. Equation (27) then gives

$$\frac{d\psi(x)}{dx} = 2k - 2\phi(x)\sin\psi(x)\,. \tag{30}$$

This gives the evolution of the phase and consequently the distribution of the nodes of the wave function $u(x)$ from which one can get the density of states (compare with equation (2.5) of [22]).

As in the previous section one may introduce $Z(x,k)$, the derivative of $\psi(x,k)$ with respect to k, which satisfies

$$\frac{dZ(x)}{dx} = 2 - 2\phi(x)Z(x)\cos\psi(x) \tag{31}$$

again the initial conditions for the two variables are $\psi(0)=\pi$ and $Z(0)=0$.

Integrating the coupled equations (30,31) between 0 and L gives the phase shift and the time delay.

4.3 The phase distribution

At this stage the formalism developed to compute the phase shift and the time delay is quite general. We now consider the case where $\phi(x)$ is white noise. Equation (30) is then a stochastic differential equation in the Stratonovich sense from which we can write down a Fokker-Planck equation for the probability density of ψ :

$$\frac{\partial P(\psi;x)}{\partial x} = -2k\frac{\partial}{\partial\psi}P(\psi;x) + 2\sigma\frac{\partial}{\partial\psi}\sin\psi\frac{\partial}{\partial\psi}\sin\psi P(\psi;x)\ . \tag{32}$$

The stationary distribution for ψ is given by:

$$P_s(\psi) = \mathcal{N}\,\frac{\mathrm{e}^{-\frac{k}{\sigma}\operatorname{cotg}\psi}}{\sin\psi}\int_\psi^\pi d\psi'\frac{\mathrm{e}^{\frac{k}{\sigma}\operatorname{cotg}\psi'}}{\sin\psi'} \tag{33}$$

where $\mathcal{N}$ is a normalization constant related to the integrated density of states per unit length $N(E) = 2kP_s(\pi) = 2\sigma\mathcal{N}$ (see equation (2.18) of [22] and reference therein). One can extract from (33) the high energy expansion of the stationary distribution:

$$P_s(\psi) = \mathcal{N}\,\frac{\sigma}{k}\left\{1 + \frac{\sigma}{2k}\sin 2\psi + \frac{\sigma^2}{k^2}\left(\frac{1}{2}\sin^2 2\psi - \sin^4\psi\right) + O\left(\frac{\sigma^3}{k^3}\right)\right\}\ . \tag{34}$$

This shows that in the high energy limit the phase is a uniformly distributed variable on $[0, 2\pi]$. This could have been guessed directly from the differential equation for ψ since in equation (30) one expects that for $k \gg \sigma$ the first term will dominate.

4.4 The time delay

Since ψ is a rapidly varying variable, one expects that the two coupled differential equations will reduce to only one, after averaging over the rapid variable. For this purpose, the next natural assumption is the decorrelation of the two variables ψ and Z in the high energy limit. We first integrate perturbatively equations (30,31) to show that this is indeed the case:

$$\psi(x) = \pi + 2kx + 2\int_0^x dx'\,\phi(x')\sin 2kx' + \cdots \tag{35}$$

Since the integral is of the order $\sqrt{\sigma x}$, this expansion is valid if $\sqrt{\sigma x} \ll 1$. The computation of the autocorrelation function $\langle\psi(x)\psi(y)\rangle - \langle\psi(x)\rangle\langle\psi(y)\rangle$ then shows that the variable ψ behaves in this limit as a Brownian motion with a drift:

$$\psi(x) \simeq \pi + 2kx + \sqrt{2}\int_0^x dx'\,\phi(x')\ . \tag{36}$$

One can perform the same approximation for Z and compute the correlation function with the help of the two expansions

$$\langle\psi(x)Z(x)\rangle - \langle\psi(x)\rangle\langle Z(x)\rangle \simeq \frac{1}{k}\left(\frac{\sigma}{4k}\sin 4kx - \sigma x\cos 4kx\right)\ . \tag{37}$$

From which we deduce that the correlations between ψ and Z will vanish in the high energy limit.

From the two stochastic differential equations (30,31) we may write the Fokker-Planck equation for the joint probability density $P(\psi, Z; x)$:

$$\frac{\partial P}{\partial x} = -2k\frac{\partial P}{\partial \psi} - 2\frac{\partial P}{\partial Z} + 2\sigma\left\{\frac{\partial}{\partial \psi}\sin\psi\frac{\partial}{\partial \psi}\sin\psi\, P + \frac{\partial}{\partial \psi}\sin\psi\frac{\partial}{\partial Z}Z\cos\psi\, P \right.$$
$$\left. + \frac{\partial}{\partial Z}Z\cos\psi\frac{\partial}{\partial \psi}\sin\psi\, P + \frac{\partial}{\partial Z}Z\cos\psi\frac{\partial}{\partial Z}Z\cos\psi\, P\right\}. \tag{38}$$

Using the fact that ψ is uniformly distributed and decorrelated from Z in the high energy limit, we may average over ψ since the joint probability density factorises as $P(\psi, Z; x) \simeq \frac{1}{2\pi}P(Z; x)$. This leads to the following Fokker-Planck equation for $P(Z; x)$:

$$\frac{\partial P(Z;x)}{\partial x} = \frac{\partial}{\partial Z}\left[(\sigma Z - 2)P(Z;x)\right] + \sigma\frac{\partial}{\partial Z}Z\frac{\partial}{\partial Z}Z\, P(Z;x) \tag{39}$$

which is the same as equation (11) provided one replaces $\frac{8k^2}{\sigma_{\rm g}}$ by $\frac{2}{\sigma}$. In particular this implies that the distribution for the variable Z is still given by equation (12) provided λ is the localization length for the supersymmetric model.

One may thus give a representation for the time delay, valid for both models:

$$\tau(k) = \frac{1}{k}\int_0^L dx\left(\mathrm{e}^{2\int_x^L dx'\,(\frac{\eta(x')}{\sqrt{\lambda}} - \frac{1}{\lambda})} - 1\right) \tag{40}$$

where $\eta(x)$ is the white noise of variance 1 entering in the potential (equal to $\frac{1}{\sqrt{\sigma_{\rm g}}}V(x)$ in the first case and $\frac{1}{\sqrt{\sigma}}\phi(x)$ in the second) and λ is the localization length.

5 Conclusion

We have shown in this paper that, for two models of random potential, the time delay exhibits the same distribution. These statistical properties are in agreement with other approaches, underlying the universality of such properties.

We have also demonstrated that in both cases the time delay may be expressed as an exponential functional of the white noise which enters in the potential.

It would be interesting to extend this approach to the multichannel case for which the stationary distribution has been recently obtained [6]. Another point that deserves attention is the fact that the resulting distribution for the time delay is exactly of the same form as the waiting time distribution that occurs in the context of classical diffusion in a random potential [14]. The supersymmetric model may help to explore this relation.

Acknowledgments

We thank Eugène Bogomolny, Pierre Le Doussal, Rodolfo Jalabert, Cécile Monthus and Marc Yor for interesting discussions. We thank Niall Wheelan for reading the manuscript.

References

[1] Büttiker M, *Electronic properties of multilayers and low-dimensional semiconductors structures*, edited by J. M. Chamberlain et al. (Plenum, New York, 1990), p. 297

[2] Landauer R and Martin T 1994 *Rev. Mod. Phys.* **66** 217

[3] Fyodorov Y V and Sommers H-J 1996 *J. Math. Phys.* **38**(4) 1918

[4] Fyodorov Y V and Sommers H-J 1996 *Phys. Rev. Lett.* **76** 4709

[5] Gopar V A, Mello P A and Büttiker M 1996 *Phys. Rev. Lett.* **77** 3005

[6] Brouwer P W, Frahm K M and Beenakker C W preprint *chao-dyn*/9705015

[7] Mucciolo E R, Jalabert R A and Pichard J-L preprint *cond-mat*/9703178

[8] Jayannavar A M,Vijayagovindan G V and Kumar N 1989 *Z. Phys. B - Cond.Mat.* **75** 77

[9] Heinrichs J 1990 *J. Phys. Condens. Matter* **2** 1559

[10] Rammal R and Doucot B 1987 *J. Physique* **48** 509

[11] Sulem P L 1973 *Physica* **70** 190

[12] Yor M 1992 *Adv. Appl. Prob.* **24** 509

[13] Schenzle A and Brand H 1979 *Phys. Rev.* **A 20** 1628

[14] Bouchaud J-P, Comtet A, Georges A and Le Doussal P 1990 *Ann. Phys. (N.Y.)* **201** 285 ; Laloux L and Le Doussal P preprint *cond-mat*/9705249 ; Le Doussal P 1987 *Thèse de Doctorat de l'Université Paris 6*

[15] Comtet A and Monthus C 1996 *J. Phys. A: Math. Gen.* **29** 1331

[16] Oshanin G, Mogutov A and Moreau M 1993 *J. Stat. Phys* **73** 379

[17] Monthus C and Comtet A 1994 *J. Phys. I (France)* **4** 635

[18] Comtet A, Monthus C and Yor M *Exponential functionals of Brownian motion and disordered systems*, to appear in *J. of Applied Prob.*, preprint *cond-mat*/9601014

[19] Faris W G and Tsay W J 1994 *SIAM J. Appl. Math.* **54** 443

[20] Carmona P, Petit F and Yor M 1994 *Stochast. Rep.* **47** 71

[21] Carmona P, Petit F and Yor M *On the distribution and asymptotic results for exponential functionals of Levy proceses*, in preparation

[22] Comtet A, Desbois J and Monthus C 1995 *Ann. Phys. (N.Y.)* **239** 312

[23] Junker G *Supersymmetric methods in quantum and statistical physics.* Springer 1996

[24] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267

[25] Fabrizio M and Mélin R preprint *cond-mat*/9701149

[26] Dunne G and Feinberg J preprint *hep-th*/9706012

[27] Ovchinnikov A A and Erikhman N S 1977 *Zh. Eksp. Teor. Fiz.* **73** 650

[28] Bouchaud J-P, Comtet A, Georges A and Le Doussal P 1987 *Europhys. Lett.* **3** 653