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Non-Hermitian β-ensemble with real eigenvalues

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By removing the Hermitian condition of the so-called β-ensemble of tridiagonal matrices, an ensemble of non-Hermitian random matrices is constructed whose eigenvalues are all real. It is shown that they belong to the class of pseudo-Hermitian operators. Its statistical properties are investigated.
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I. INTRODUCTION

About a decade ago, evidences have been gathered that complex Hamiltonian may have real spectrum.[1] It has also been identified that this property is a consequence of the invariance of the Hamiltonian under the combined parity and time-reversal transformations. This understanding lead to an extension of quantum mechanics to include this special class of non-Hermitian Hamiltonians with unbroken *PT* symmetry[2] (see[3] for a review). From a more general approach, the main property of this class of non-Hermitian operators is to be connected to their respective adjoints by the similarity transformation

$$H^{\dagger} = \eta H \eta^{-1} \tag{1}$$

which defines the so-called pseudo-Hermitian operators and means that operator and its adjoint have the same set of eigenvalues.[4] Concomitantly, there has been attempts to find, in the context of random matrix theory (RMT), ensembles of matrices satisfying the above relation.[5,6] This effort produced, so far, an ensemble of 2×2 matrices, the case of arbitrary size matrices remaining open.

Non-Hermitian random matrices were introduced by Ginibre[7] few years after Wigner's proposal of the Hermitian ones by the end of the 50's.[8] He just removed the Hermitian condition from the three classes of matrices of the Gaussian ensemble, namely the orthogonal (GOE), the unitary (GUE), and the symplectic (GSE) with real, complex, and quaternion elements respectively. Denoted by Dyson index $\beta = 1, 2, 4$ which gives the degree of freedom of the number, these classes constitute what Dyson called the "three-fold way".[9] The β-ensemble generalized this "three-fold way" by constructing a β dependent ensemble of tridiagonal Hermitian matrices in which β is a parameter that can assume any positive real value.[10,11] For the integer 1, 2, 4 values of β, the statistical properties of the Gaussian ensemble[12] are reproduced.

Ginibre's non-Hermitian classes of matrices have found applications in the study of physical open systems[13] and have been matter of recent investigation.[14] However, they do not constitute good candidates to the construction of a pseudo-Hermitian RMT. It is unlikely that by imposing some restriction on their matrices, they can be made to satisfy Eq. (1). It is our purpose to show that, on the other hand, by removing the Hermitian condition of the tridiagonal matrices, the β-ensemble naturally extends RMT to include the pseudo-Hermiticity condition. This is shown below in the third section, after the presentation of the β-ensemble in the next section.

II. THE HERMITIAN β-ENSEMBLE

For completeness and future comparison we give in this section a summary of some basic results regarding the β-ensemble. A matrix of this ensemble is[11]

$$H_\beta = \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & & \chi_\beta & N(0,2) \end{pmatrix} \tag{2}$$

where the diagonal elements are normally distributed and the sub-diagonal elements $\chi_\nu/\sqrt{2}$ are distributed according to the distribution

$$f_\nu(y) = \frac{2\exp(-y^2)y^{\nu-1}}{\Gamma[\nu/2]} \tag{3}$$

derived from the χ^2-distribution. From this definition, it is found that the joint density distribution of the eigenvalues is given by[11]

$$P(x_1, x_2, ..., x_n) = C_n \exp\left(-\frac{1}{2}\sum_{k=1}^{n} x_k^2\right) \prod_{j>i} |x_j - x_i|^\beta. \tag{4}$$

Although some results have already been obtained,[15,16] analytical derivations of exact expressions describing eigenvalue statistical properties for arbitrary values of β are still lacking. Results, however, can be inferred in the two extreme limits $\beta \to 0$ and $\beta \to \infty$ and, also, in the asymptotic regime of large matrices. For instance, since

$$\lim_{\nu\to 0} f_\nu(y) = 2\delta(y), \tag{5}$$

H_0 becomes diagonal with eigenvalues behaving independently. Therefore, in this limit, the β-ensemble becomes a Poissonian ensemble of uncorrelated levels. Regarding the infinite limit, as the two first moments of the distribution $f_\nu(y)$ are

$$\langle y \rangle = \frac{\Gamma[(\nu+1)/2]}{\Gamma(\nu/2)} \xrightarrow{\nu \gg 1} \sqrt{\frac{\nu}{2}} \tag{6}$$

and

$$\langle y^2 \rangle = \frac{\nu}{2}, \tag{7}$$

it follows that for large β, that is large ν, the average $\langle y \rangle$ diverges while the variance vanishes. Under this circunstance, the diagonal elements of H_β becomes negligible compared to the sub-diagonals ones and the fluctuations are progressively supressed turning rigid the ensemble.

Asymptotically, as the size n of the matrices increases, the eigenvalues of H_β distribute themselves on the real axis with the same density the Gaussian ensemble.[11] This means they occupy, as proved below, a compact support defined by Wigner semi-circle law[12]

$$\rho(x) = \frac{1}{\pi\beta}\sqrt{2n\beta - x^2}. \tag{8}$$

Finally, we consider the case of matrix of size $n = 2$ which analytically can be fully worked out. The important measure is the density distribution of the distance s between the couple of eigenvalues. It is given by

$$P(t) = \int \frac{da_1 da_2 dy}{2\pi} \exp\left(-\frac{a_1^2}{2} - \frac{a_2^2}{2}\right) f_\beta(y)\delta\left[t - \sqrt{\left(\frac{a_1 - a_2}{2}\right)^2 + y^2}\right] \tag{9}$$

which, after a simple calculation, leads to

$$P(t) = \frac{2}{\Gamma[(\beta+1)/2]} t^{\beta} \exp(-t^2). \tag{10}$$

After the rescaling, $s = t/\langle t\rangle$, of the variable such that $\langle s\rangle = 1$, the distribution $P(s)$ with

$$\langle t\rangle = \frac{\Gamma(1+\beta/2)}{\Gamma[(1+\beta)/2]} \tag{11}$$

that generalizes the so-called Wigner surmise is obtained. In agreement with the above discussion, $P(s)$ has the limits

$$P(s) = \frac{2}{\pi} \exp\left(-\frac{s^2}{\pi}\right) \tag{12}$$

when $\beta \to 0$ and when $\beta \to \infty$

$$P(s) = \delta(s-1). \tag{13}$$

The distribution $P(s)$ fit reasonably well the spacing between eigenvalues of large matrices.

III. THE PSEUDO-HERMITIAN β-ENSEMBLE

Considering the general case of a non-Hermitian tridiagonal matrix, H, with diagonal $a = (a_n, ..., a_1)$, upper sub-diagonal $b = (b_{n-1}, ..., b_1)$, and lower sub-diagonal $b = (c_{n-1}, ..., c_1)$, in which sub-diagonal elements are different from zero but otherwise matrix elements can assume any real value, then two theorems are easily proved:

Theorem 1. *The matrix H is pseudo-Hermitian.*

Proof. Define the invertible diagonal matrix η whose elements are given by

$$\mathrm{diag}(\eta) = \left(1, \frac{b_{n-1}}{c_{n-1}}, \frac{b_{n-1}b_{n-2}}{c_{n-1}c_{n-2}}, ..., \frac{b_{n-1}b_{n-2}...b_1}{c_{n-1}c_{n-2}...c_1}\right) \tag{14}$$

then, it is immediately verified that H and its adjoint $H^{\dagger}$ satisfy Eq. (1) belonging therefore to the class of pseudo-Hermitian matrices. □

Theorem 2. *If the products $b_i c_i$ are positive then all eigenvalues of H are real.*

Proof. Define the invertible diagonal matrix $\eta^{\frac{1}{2}}$ whose elements are obtained by taking the square roots of the elements of η, that is

$$\mathrm{diag}(\eta^{\frac{1}{2}}) = \left(1, \sqrt{\frac{b_{n-1}}{c_{n-1}}}, \sqrt{\frac{b_{n-1}b_{n-2}}{c_{n-1}c_{n-2}}}, ..., \sqrt{\frac{b_{n-1}b_{n-2}...b_1}{c_{n-1}c_{n-2}...c_1}}\right) \tag{15}$$

then, the matrix

$$K = \eta^{\frac{1}{2}} H \eta^{-\frac{1}{2}} = \eta^{-\frac{1}{2}} \left(\eta H \eta^{-1}\right) \eta^{\frac{1}{2}} = \eta^{-\frac{1}{2}} H^{\dagger} \eta^{\frac{1}{2}} = K^{\dagger} \tag{16}$$

is Hermitian with diagonal a and sub-diagonal $\left(\sqrt{b_{n-1}c_{n-1}}, ..., \sqrt{b_1 c_1}\right)$. □

Let us now introduce the random non-Hermitian matrix $\hat{H}_{\beta}$ obtained from H_{β} by allowing the elements of the two sub-diagonals to be different though sorted from the same $f_{(n-i)\beta}(y)$ distribution. In this case, the probability of matrices with zero elements is negligible, and, from the above theorems, $\hat{H}_{\beta}$ is a pseudo-Hermitian matrix and its eigenvalues are real. Moreover, the eigenvalues of $\hat{H}_\beta$ are the same of the matrix

$$K_\beta = \eta^{\frac{1}{2}} \hat{H}_\beta \eta^{-\frac{1}{2}} = \begin{pmatrix} N(0,1) & \kappa_{(n-1)\beta} & & & & \\ \kappa_{(n-1)\beta} & N(0,1) & \kappa_{(n-2)\beta} & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & \kappa_{2\beta} & N(0,1) & \kappa_\beta \\ & & & & \kappa_\beta & N(0,1) \end{pmatrix} \tag{17}$$

where κ_ν are distributed according to the K-distribution

$$g_\nu(z) = \frac{8}{\Gamma^2(\nu/2)} z^{2\nu-1} K_0\left(2z^2\right), \tag{18}$$

where $K_0(x)$ is the modified Bessel function. This distribution has the limit

$$\lim_{\nu\to 0} g_\nu(z) = 2\delta(z) \tag{19}$$

such that $\hat{H}_0$ coincides with H_0 when $\beta \to 0$. In the other extreme, the moments

$$\langle z \rangle = \left[\frac{\Gamma[(2\nu+1)/4]}{\Gamma(\nu/2)}\right]^2 \xrightarrow{\nu \gg 1} \sqrt{\frac{\nu}{2}} \tag{20}$$

and

$$\langle z^2 \rangle = \left[\frac{\Gamma[(\nu+1)/2]}{\Gamma(\nu/2)}\right]^2 \xrightarrow{\nu \gg 1} \frac{\nu}{2} \tag{21}$$

implies that $\hat{H}_\beta$ also becomes rigid when $\beta \to \infty$. Therefore, the non-Hermitian and the Hermitian beta ensembles coincide in the two extreme limits.

For large matrix sizes, the eigenvalues of the matrices K_β, and *a fortiori* those of $\hat{H}_\beta$, are asymptotically distributed on the real axis as those of H_β, namely according to the Wigner semi-circle, Eq. (8). To prove this, we resort to the fact that the characteristic polynomials of the general tridiagonal matrix H satisfy, with $P_{-1} = 0$ and $P_0 = 1$, the recurrence relation

$$P_n(x) = (a_n - x)P_{n-1}(x) - b_{n-1}c_{n-1}P_{n-2}(x). \tag{22}$$

In particular, applying to H_β and K_β, the independence of their elements can be used to define an average characteristic polynomial $\langle P_n \rangle$ constructed by taking the average

$$\langle P_n \rangle = (\langle a_n \rangle - x)\langle P_{n-1} \rangle - \langle b_{n-1}c_{n-1} \rangle \langle P_{n-2} \rangle, \tag{23}$$

of Eq. (22). Considering first the Hermitian case, H_β, $\langle a_n \rangle = 0$ and

$$\langle b_{n-1}c_{n-1} \rangle = \left\langle \left[\frac{\chi_{(n-1)\beta}}{2}\right]^2 \right\rangle = \frac{(n-1)}{2}\beta \tag{24}$$

which replaced in Eq. (23) gives

$$\langle P_n \rangle = -x\langle P_{n-1} \rangle - \frac{n-1}{2}\beta \langle P_{n-2} \rangle. \tag{25}$$

Comparing this relation with the recurrence relation of the Hermite polynomials, the identification

$$\langle P_n \rangle = \left(\frac{\beta}{2}\right)^{n/2} H_n\left(-\frac{x}{\sqrt{2\beta}}\right). \tag{26}$$

follows.[11] Therefore, eigenvalues of H_β fluctuate around the zeros of the Hermite polynomials which are known to be distributed according to Eq. (8).[17] Considering now the non-Hermitian case, for large n we have

$$\langle b_{n-1}c_{n-1} \rangle = \langle \kappa^2_{(n-1)\beta} \rangle = \left[\frac{\Gamma\left(\frac{n-1}{2}\beta + \frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\beta\right)}\right]^2 \sim \frac{n-1}{2}\beta \tag{27}$$

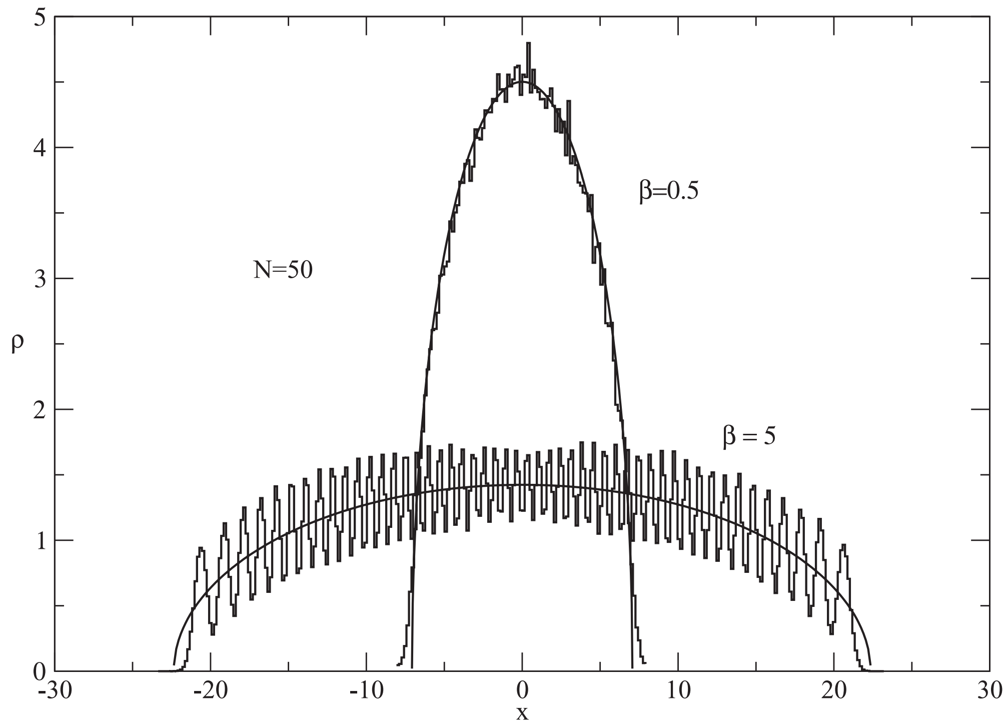

FIG. 1. For $\beta = 0.5$ and $\beta = 5$, the density distribution of eigenvalues of an ensemble of non-Hermitian matrices of size $N = 50$ is compared with the semi-circle law prediction.

implying the same eigenvalue asymptotic density distribution of the Hermitian case, namely Eq. (8). Therefore, although differences are to be expected in their fluctuations properties, the eigenvalues of the tridiagonal matrices of the non-Hermitian β-ensemble, asymptotically occupy, on the real axis, the same compact interval of the Hermitian ones. Though the above argument mathematically is not rigorous, the correctness of its prediction is shown in Fig. 1, where the eigenvalue density distribution generated by an ensemble of matrices of size $N = 50$ is compared with the semi-circle law. The histogram for $\beta = 5$ shows that at this relatively high value of the parameter, the eigenvalues place themselves in a crystal lattice structure fluctuating weakly around average positions.

Considering the 2×2 case, replacing in Eq. (9) the distribution $f_\beta(y)$ by $g_\beta(y)$, the spacing distribution between the eigenvalues

$$P(t) = \frac{t^{2\beta} e^{-\frac{t^2}{4}}}{\Gamma^2(\beta/2)4^{\beta-1}\sqrt{\pi}} \int_0^1 \frac{dv}{\sqrt{1-v}} \exp\left(\frac{t^2 v}{4}\right) v^{\beta-1} K_0\left(\frac{t^2 v}{2}\right). \tag{28}$$

is obtained. As before, a new variable s is defined by the rescaling which ensures that $\langle s \rangle = 1$. Replacing in Eq. (28) the Bessel function by its behavior $-\log(x)$ at the origin we find that, at small separations, the eigenvalues repel one another as

$$P(s) \sim -s^{2\beta} \log(s). \tag{29}$$

We remark that logarithmic dependent repulsion of pseudo-Hermitian matrices has already been reported.[18] On the other hand, for large values of s, making in integral the change of variable $v = 2u/t^2$ we find that P(s) decays exponentially as

$$P(s) \sim e^{-\frac{1}{4}\langle t \rangle^2 s^2} \tag{30}$$

without the power factor of the Hermitian case, Eq. (10). By comparing Eqs. (10) and (29), we conclude that the repulsion between eigenvalues becomes larger when the Hermitian condition is removed and, at the same time, the absence of the power factor in Eq. (30) shows that

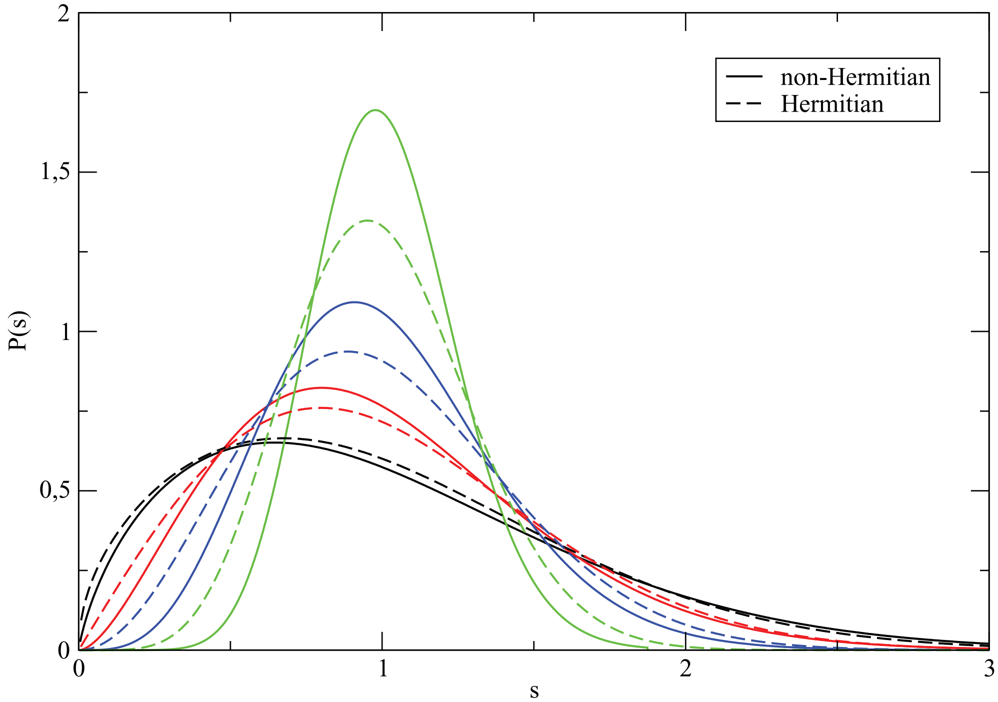

FIG. 2. The spacing distributions calculated with Eqs. (10) (dashed lines) and (28) (full lines), are compared for the values 0.5 (black), 1 (red), 2 (blue), and 5 (green) of β.

larger separations become less probable. In Fig. 2, a set of spacing distributions calculated with Eqs. (10) and (28) illustrates this discussion. They show the evolution of the spacings towards the Gaussian distribution when β decreases and the tendency for a more localized distribution as beta reaches higher values. Fig. 2 shows that the non-Hermitian ensemble moves faster in this direction explaining the structure in the eigenvalue density for the value $\beta = 5$ in Fig. 1.

In conclusion, we have shown that by an appropriate removal of the Hermitian condition, the β-ensemble of tridiagonal matrices becomes a model of pseudo-Hermitian matrices with real eigenvalues. This extension of the β-ensemble parallels in RMT the extension of quantum mechanics to incorporate complex Hamiltonian with real eigenvalues.

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