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# SPECIAL SECTION ON CONSTRAINT SATISFACTION PROBLEMS AND MESSAGE PASSING ALGORITHMS

Recent years have seen the emergence of a fascinating cross-disciplinary area at the interface of statistical physics, combinatorial optimization, information theory, and computer science. Many important questions in various branches of science can be formulated as problems of optimization or inference, and the framework of graphical models provides a common language. Topics range from constraint satisfaction problems to error-correcting codes, spin glasses, and network reconstruction and inference in computational biology. Fundamental questions that cross the frontiers between disciplines involve the understanding and analysis of the dynamics (natural dynamics of spins or dynamics of algorithms) and the relevance of the geometrical structure of the space of solutions on this dynamics. In particular, the existence of some clustering of the solutions, a key notion in spin glass physics, has been demonstrated in several constraint satisfaction problems, giving some insight into the algorithmic hardness of constraint satisfaction and of reconstruction problems. In this context, message passing algorithms such as belief propagation and survey propagation have shown some surprisingly good performance and as such call for the elaboration of a full mathematical framework.

This volume gathers a number of contributions from some of the leading scientists in the field addressing some of these issues. It reflects the diversity of approaches and of problems, together with the importance of some basic concepts and questions, such as the structure of the space of solutions and its relation with message passing. It also shows at work truly interdisciplinary research in which each discipline finds in its core some fundamental questions, and sometimes some techniques, which turn out to be also of crucial importance to other fields.

The *geometry* of the solution space in constraint satisfaction problems is examined in at least three articles. In "On the Diameter of the Set of Satisfying Assignments in Random Satisfiable $k$-CNF Formulas," Uriel Feige, Abraham Flaxman, and Dan Vilenchik investigate random satisfiable $k$-CNF formulas and show that a small change of the density of constraints near the satisfiability threshold of random formulas has a huge impact on the diameter of the solution space. The paper "Quiet Planting in the Locked Constraint Satisfaction Problems," by Lenka Zdeborová and Florent Krzakala, also studies satisfiable ("planted") instances of constraint satisfaction problems: when the density of constraints increases, the solution of such a problem goes from an easy phase to a hard phase with many solutions to a hard phase with only one solution (the planted one) and finally becomes easy, with only the planted solution. The physical "cavity method" provides a heuristic, but presumably exact, estimate of these various thresholds. Developing corresponding rigorous approaches along these lines is a challenging research program in the field.

Contributing to such general understanding of the geometry of the solution space of random CSPs (particularly in the satisfiable regime) and further providing rigorous foundation for the statistical mechanics heuristics are the primary objectives of the work reported in "Reconstruction and Clustering in Random Constraint Satisfaction Problems" by Andrea Montanari, Ricardo Restrepo, and Prasad Tetali. One of the motivations for the study of the so-called reconstruction problem is the fact that solutions within a cluster impose long-range correlations among assignments of variables. By considering binary variable CSPs (with $k$-ary constraints) whose factor graph is a random tree, these authors prove bounds on the reconstruction and clustering thresholds which coincide up to the first order as $k$ goes to infinity. They also verify a sufficient condition,



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provided in an earlier work of Gerschenfeld and Montanari, to transfer reconstruction results (on binary CSPs as well as colorings of sparse random graphs) from trees to the same on *sparse* random graphs, further validating the general methodology that questions on the geometry of the solution space of sparse random CSPs often reduce to tree calculations.

The study of (non)reconstruction thresholds is also motivated by its link to the limit of efficiency of Glauber dynamics to sample from the set of solutions to a CSP. Toward this, strong forms of nonreconstruction results are reported in "Reconstruction for Colorings on Trees," by Nayantara Bhatnagar, Juan Vera, Eric Vigoda, and Dror Weitz, for the (asymptotically optimal) regime of $Cb/\ln b$ colors, with $C > 1$, for regular $b$-branching trees. The authors demonstrate that their results are strong enough, in turn, to yield fast mixing of local *constant-size block dynamics* on the space of colorings. Using a variation of the standard block dynamics and weighted canonical path analyses, tight upper and lower bounds on the mixing time of *Glauber dynamics* on the space of colorings of trees are provided in "The Glauber Dynamics for Colorings of Bounded Degree Trees" by Brendan Lucier and Michael Molloy.

Circumventing the difficulty in explicitly computing marginals at the root of a tree, a novel and computationally efficient notion of a "survey" of the distribution is introduced in "A Computational Method for Bounding the Probability of Reconstruction on Trees" by Nayantara Bhatnagar and Elitza Maneva, and bounds on the reconstruction threshold for the Potts model on small-degree trees are provided as a concrete application.

Extending the applicability of the reconstruction techniques to a biological setting, in "Phylogenies without Branch Bounds: Contracting the Short, Pruning the Deep," authors Constantinos Daskalakis, Elchanan Mossel, and Sebastien Roch introduce a new and efficient distance-based phylogenetic reconstruction algorithm with few assumptions. Motivated by the so-called multicasting models of communication networks, in "The Multistate Hardcore Model on a Regular Tree," authors David Galvin, Fabio Martinelli, Kavita Ramanan, and Prasad Tetali consider, and provide rigorous bounds for, the classical questions of phase uniqueness and coexistence in a generalized hardcore model with integer capacity $C \geq 1$; confirming earlier conjectures of Mitra et al., the authors demonstrate quantitative dependence of the nature of the phase transition on the *parity* of $C$.

Constraint satisfaction problems form an important and robust enough model to make the study of partial solutions to instances (as in satisfying only a fraction of constraints) challenging and interesting. Inspired by this, in "Limit Behavior of Locally Consistent Constraint Satisfaction Problems," the authors Manuel Bodirsky and Daniel Král' study notions of *local consistency*, as dictated by $k$-sets of constraints or alternately all constraints involving $k$-sets of variables, of CSP instances. The authors study the effect of such local consistency on the quality of global optimal solutions, with respect to the (maximum) number of constraints that can be simultaneously satisfied.

In "Maximum Weight Partial Colorings on Sparse Random Graphs," Steven Jaslar and Sekhar Tatikonda develop the local weak convergence method for the problem of random graph coloring, a graphical model where the variables are not binary. The basic challenge addressed here is the control of long-range correlations. This same kind of control is needed in a totally different problem, that of optimal decoding in error correcting codes based on low-density graphs. In "Decay of Correlations for Sparse Graph Error Correcting Codes," Shrinivas Kudekar and Nicolas Macris show, using statistical mechanics methods, that the correlations decay fast enough in this problem when the

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noise is small enough: in this regime the very fast belief-propagation decoding gives the optimal result.

The use of message passing algorithms, and in particular belief propagation, is the subject of several other contributions. The paper "Belief Propagation for Weighted b-Matchings on Arbitrary Graphs and Its Relation to Linear Programs with Integer Solutions," by Mohsen Bayati, Christian Borgs, Jennifer Chayes, and Riccardo Zecchina, shows the deep links of the belief propagation convergence with linear programming relaxation, in the case of weighted matching problems. Interestingly, this is one of the rare cases where the belief propagation performance can be assessed on a general graph without any constraint on the local tree-like structure. In "Counting Independent Sets Using the Bethe Approximation," Venkat Chandrasekaran, Misha Chertkov, David Gamarnik, Devavrat Shah, and Jinwoo Shin show a belief propagation–type algorithm that is guaranteed to converge to a stationary point of the free energy, and they use it to estimate the number of independent set in a graph.

In summary, this collection of thirteen papers provides a sampling of several important directions of current interest in the topic of constraint satisfaction problems and message passing algorithms.

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