

Persistence of a continuous stochastic process with discrete-time sampling

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We introduce the concept of “discrete-time persistence,” which deals with zero-crossings of a continuous stochastic process, $X(T)$, measured at discrete times, $T = n\Delta T$. For a Gaussian Markov process with relaxation rate μ , we show that the persistence (no crossing) probability decays as $[\rho(a)]^n$ for large n , where $a = \exp(-\mu\Delta T)$, and we compute $\rho(a)$ to high precision. We also define the concept of “alternating persistence,” which corresponds to $a < 0$. For $a > 1$, corresponding to motion in an unstable potential ($\mu < 0$), there is a nonzero probability of having no zero-crossings in infinite time, and we show how to calculate it.

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Persistence of a continuous stochastic process has generated much recent interest in a wide variety of nonequilibrium systems including various models of phase ordering kinetics, diffusion, fluctuating interfaces, and reaction-diffusion processes [1]. Persistence has also been recently used in fields as diverse as ecology [2] and seismology [3]. Persistence is simply the probability $P(t)$ that a stochastic process $x(t)$ does not change sign up to time t . In most of the systems mentioned above, $P(t) \sim t^{-\theta}$ for large t , where the persistence exponent θ is nontrivial. Apart from various analytical and numerical results, this exponent has also been measured experimentally in systems such as breath figures [4], liquid crystals [5], soap bubbles [6], and more recently in laser-polarized Xe gas using NMR techniques [7].

Persistence has also remained a popular subject among applied mathematicians for many decades [8]. They are most interested in the probability of “no zero crossing” of a Gaussian *stationary* process (GSP) between times T_1 and T_2 [9]. It is well known that this probability usually decays as $\sim \exp(-\theta T)$ for large $T = |T_2 - T_1|$, where θ is nontrivial [9,8]. The persistence of some of the *nonstationary* processes mentioned in the preceding paragraph, such as the diffusion processes, can be mapped to that of a corresponding GSP [10]. This makes the two sets of problems related to each other and the power law exponent in the former problem becomes the inverse decay rate in the latter. Even though θ is, in general, hard to compute analytically, it is very easy to evaluate numerically in most cases. Given this fact, and the combined interest of both statistical physicists and applied mathematicians, much recent effort has been devoted to computing θ numerically to extremely high precision.

This raises a natural question: How accurately can one measure θ ? Is there a natural limitation and if so, can it be overcome? This issue arises from the following simple observation. All the stochastic processes mentioned above occur in continuous time. However, when one performs numerical simulations or experiments on persistence, one has to discretize time in some way and sample the data only at these discrete time points to check if the process has retained its sign. Due to this discretization, some information is lost. For example, the process may have crossed and recrossed zero (or a spin may have flipped sign many times) between

two consecutive discrete time points. These crossings (or sign flips) go undetected due to the discrete sampling of the data. The question is how serious is this loss of information. Is it possible to estimate quantitatively the error involved due to the discretization?

The purpose of this Rapid Communication is twofold: (i) to point out that there is indeed a very general and nontrivial effect, due to the discretization of time, on the measured persistence of any continuous stochastic process, and (ii) to provide a quantitative estimate of its magnitude in a simple Markov model. The effect turns out to be nontrivial even for this simple toy model. We also develop two new analytical approaches, perturbative and variational, which provide results to extremely high precision. We emphasize that, even though we restrict ourselves here to a simple model by way of an example, this effect is very general and should be observable in simulations or experiments on more realistic systems.

To formulate a precise quantitative question, let us consider a stationary stochastic process in continuous time T which is sampled at times $T_1, T_2, \dots, T_n = T$ separated by a uniform window size, $T_i - T_{i-1} = \Delta T$ such that $T = n\Delta T$. The continuous persistence $P(T)$ is then approximated as $P(T) \approx P_n$, where P_n is the probability that the process $X(T)$ is positive at all the n discrete points. Note that, for finite ΔT , P_n is different from $P(T)$ since the process can cross zero more than once between two successive discrete times. One expects that the approximation $P(T) \approx P_n$ will improve as the window size, ΔT , decreases, and in the limit $\Delta T \rightarrow 0$, $n \rightarrow \infty$ keeping $T = n\Delta T$ fixed, $P_n \rightarrow P(T)$. By contrast, if the window size $\Delta T \gg \tau$ where τ is the correlation time of the process, the stochastic variables at different discrete points become completely uncorrelated and we expect $P_n \rightarrow 2^{-n}$, since the probability that at each point the process is positive is just 1/2. We then ask: How does the discrete persistence P_n interpolate between these two limits as ΔT varies continuously from 0 to ∞ ? We show that for a GSP, in general, $P_n \sim [\rho(\Delta T)]^n$ for large n , where the function $\rho(\Delta T)$ is nontrivial with the limiting behavior

$$\rho(\Delta T) \approx \begin{cases} 1 - \theta\Delta T, & \Delta T \rightarrow 0 \\ 1/2, & \Delta T \rightarrow \infty, \end{cases} \quad (1)$$

where θ is the usual persistence exponent. As $\Delta T \rightarrow 0$, one recovers the continuous persistence, $P_n \rightarrow (1 - \theta \Delta T)^n \sim \exp(-\theta T)$ where $T = n \Delta T$. The general goal would be to compute this function $\rho(\Delta T)$, the knowledge of which will provide an estimate of the difference, due to the finite window size ΔT , between the measured persistence P_n and the $P(T)$ of the underlying continuous process.

The nonstationary processes discussed in the first paragraph are related to the equivalent stationary ones via $T = \ln t$ [10]. A uniform spacing, ΔT , between measurements in the latter systems, therefore, corresponds to measurements uniformly spaced in *log time* in the former. Such a measurement regime has indeed been used in a recent experimental study of diffusive persistence [7], with a spacing in log-time equivalent to $\Delta T \approx 0.24$. The present paper is the first step in understanding how such discretization affects the measured result. To compare directly with the experiment, we need to compute the function $\rho(\Delta T)$ for the diffusion equation which is hard due to the non-Markovian nature of the process. However, to understand the general nature of this function $\rho(\Delta T)$, it would be useful to find a toy model where it can be computed explicitly. We consider below a simple Gaussian Markov process for which progress can be made in that direction. The physical process we study is the one-dimensional Ornstein-Uhlenbeck motion of a noisy, overdamped particle in a potential $V(X) = \mu X^2/2$, where the position X of the particle evolves via the Langevin equation,

$$\frac{dX}{dT} = -\mu X + \eta(T). \quad (2)$$

The white noise $\eta(T)$ has zero mean and a correlator $\langle \eta(T) \eta(T') \rangle = 2D \delta(T - T')$.

For this process, we first evaluate the continuous persistence and then compute the function $\rho(\Delta T)$. For the continuous persistence, a backward Fokker-Planck (BFP) approach is useful. Let $Q(X, T)$ denote the probability that, starting at X at $T=0$, the particle has not crossed the origin, $X=0$, up to time T . We expect different behavior depending on whether $\mu > 0$ (stable potential) or $\mu < 0$ (unstable potential). In the former case, the particle will eventually cross the origin and hence $Q(X, T)$ will decay exponentially with time. In the latter case, however, the particle has a finite probability to escape to infinity, and hence persistence should decay to a nonzero number. The latter case is also related to the problems of escape from metastable states studied before [11].

The probability $Q(X, T)$ satisfies the BFP equation,

$$\frac{\partial Q}{\partial T} = D \frac{\partial^2 Q}{\partial X^2} - \mu X \frac{\partial Q}{\partial X}, \quad (3)$$

with boundary conditions $Q(0, T) = 0$ and $Q(\infty, T) = 1$ for all T , and initial condition $Q(X, 0) = 1$ for all $X > 0$. The solution is

$$Q(X, T) = \operatorname{erf} \left[\frac{e^{-\mu T}}{\sqrt{2D'(1 - e^{-2\mu T})}} X \right], \quad (4)$$

where $D' = D/\mu$ and $\operatorname{erf}[x]$ is the error function. For $\mu > 0$, $Q(X, T)$ becomes separable in X and T for large

T , $Q(X, T) \sim e^{-\mu T} X$, and decays exponentially with T for fixed X . This gives the persistence exponent $\theta = \mu$. For $\mu < 0$, however, $Q(X, T)$ approaches the steady state solution $Q(X) = \operatorname{erf}[X/\sqrt{2|D'|}]$ as $T \rightarrow \infty$. We also note from Eq. (2) that the critical case $\mu = 0$ corresponds to ordinary Brownian motion, and taking the limit $\mu \rightarrow 0$ in Eq. (4), one recovers the known result, $Q(X, T) = \operatorname{erf}[X/\sqrt{4DT}]$, which decays as a power law, $Q(X, T) \sim X/\sqrt{T}$, for large T .

For later purposes, we will also need the Green's function $G(X_2, T_2 | X_1, T_1)$, the probability that the particle starting at $X = X_1$ at $T = T_1$ will reach X_2 at T_2 , with $T_2 > T_1$. This propagator can be easily computed exactly from Eq. (2) and we get,

$$G(X_2, T_2 | X_1, T_1) = \frac{1}{\sqrt{2\pi D'(1 - a^2)}} e^{-[(X_2 - aX_1)^2/2D'(1 - a^2)]}, \quad (5)$$

where $a = e^{-\mu(T_2 - T_1)}$. Note that for $\mu \geq 0$, $0 \leq a \leq 1$, while for $\mu < 0$, $a > 1$ (and $D' = D/\mu < 0$).

We now turn to the discrete persistence P_n of the continuous process in Eq. (2). Let $Q_n(X)$ be the probability that starting at X at $T=0$, the process is positive at all the discrete points $T_1 = \Delta T$, $T_2 = 2\Delta T$, \dots , $T_n = n\Delta T$ separated by the uniform window size ΔT . Then the discrete persistence is $P_n = \int_0^\infty Q_n(X) P_0(X) dX$, where $P_0(X)$ is the distribution of the initial position of the particle and can be arbitrary. Using the Markov property of the process in Eq. (2), it is easy to write down a recurrence relation for $Q_n(X)$,

$$Q_{n+1}(X) = \int_0^\infty G(Y, \Delta T | X, 0) Q_n(Y) dY, \quad (6)$$

where G is the propagator as in Eq. (5) with $a = e^{-\mu \Delta T}$ and $Q_0(X) = 1$ for all $X > 0$. This recurrence is the discrete analog of the continuous BFP equation (3). Indeed, it can be checked that Eq. (6) reduces to Eq. (3) in the limit $\Delta T \rightarrow 0$. To simplify the algebra, we consider the rescaled variable, $x = X/\sqrt{D'(1 - a^2)}$, in terms of which the recursion reads

$$Q_{n+1}(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp[-(y - ax)^2/2] Q_n(y) dy, \quad (7)$$

where we have used the explicit expression for G from Eq. (5).

Let us first consider the case $\mu > 0$, i.e., $0 \leq a = e^{-\mu \Delta T} < 1$, where, guided by the continuous case, we expect $Q_n(x) \rightarrow \rho^n q(x)$ as $n \rightarrow \infty$ at any fixed x . Substituting this asymptotic form into Eq. (7), we get an integral-eigenvalue equation for $q(x)$,

$$\rho q(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp[-(y - ax)^2/2] q(y) dy, \quad (8)$$

with eigenvalue $\rho(a)$ that evidently depends continuously on a . Although Eq. (8) admits many eigenvalues, we are interested only in the largest eigenvalue since it dominates the asymptotic behavior of $Q_n(x)$ for large n . We also note that Eq. (8) determines the eigenfunction $q(x)$ only up to an

overall multiplicative constant. Let us first consider the limit $a \rightarrow 0$ or equivalently $\Delta T \rightarrow \infty$. In this case, Eq. (8) can be solved exactly to give $\rho = 1/2$ and $q(x) = \text{const}$, thus recovering the correct limiting behavior, $Q_n(x) \rightarrow \text{const } 2^{-n}$ [and the initial condition, $Q_0(x) = 1$ for $x > 0$, fixes the constant at unity]. For small a , by expanding Eq. (8) in a Taylor series, it is easy to compute $\rho(a)$ perturbatively, giving $\rho = 1/2 + (1/\pi)a + O(a^2)$. The goal now is to evaluate $\rho(a)$ for arbitrary a . To this end we develop below two analytical approaches and compare them with the direct numerical integration of Eq. (7).

Perturbative approach. We expand the factor $\exp(axy)$, from the exponential in Eq. (8), as a power series and integrate term by term, to get

$$\rho q(x) = \frac{\exp(-a^2 x^2/2)}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{b_n}{\sqrt{n!}} (\sqrt{ax})^n, \quad (9)$$

$$b_n = \frac{a^{n/2}}{\sqrt{n!}} \int_0^{\infty} dy y^n \exp(-y^2/2) q(y). \quad (10)$$

Substituting Eq. (9) into Eq. (10) leads to the matrix eigenvalue equation

$$\rho b_n = \sum_{m=0}^{\infty} A_{nm} b_m, \quad (11)$$

$$A_{nm} = \frac{1}{\sqrt{4\pi(1+a^2)}} \left(\frac{2a}{1+a^2} \right)^{(n+m)/2} \frac{\Gamma\left(\frac{n+m+1}{2}\right)}{\sqrt{n!m!}}. \quad (12)$$

This approach converts an integral eigenvalue equation into a matrix eigenvalue equation, with matrix elements that decrease exponentially as n and m increase. Computing the largest eigenvalue of the $N \times N$ submatrix ($n, m = 0, 1, \dots, N-1$) gives a rapidly converging series of estimates for ρ as N increases. For a given N , the result is exact to order ϵ^{N-1} , where $\epsilon = 2a/(1+a^2)$. In this way one can easily obtain results for $\rho(a)$ correct to one part in 10^{12} . Convergence becomes progressively slower as $a \rightarrow 1$, which is expected since $\epsilon \rightarrow 1$ in this limit. For $a \rightarrow 1$, however, we have the analytical result $\rho \rightarrow a$ [such that $\rho^n \rightarrow \exp(-n\mu\Delta T) = \exp(-\mu T)$], since we must recover the continuum result in this limit.

Variational approach. It is possible to derive a useful variational inequality for ρ . First we note that the integral operator in Eq. (8), asymmetric in x and y , can be made self-adjoint via the substitution, $q(x) = g(x) \exp[(1-a^2)x^2/4]$ which gives,

$$\rho g(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} K(x,y) g(y) dy, \quad (13)$$

where $K(x,y) = K(y,x) = \exp\{-[(1+a^2)/4](x^2+y^2) + axy\}$. Let $f(x)$ be any normalizable function, $\int_0^{\infty} f^2(x) dx = 1$. Using elementary properties of linear vector spaces and the

TABLE I. Estimates of the eigenvalue $\rho(a)$ for $-1 \leq a \leq 1$, from numerical, variational, and perturbative methods. The latter is the most precise, being accurate to the number of figures quoted.

a	ρ_{num}	ρ_{var}	ρ_{pert}
1.0	1	1	1
0.8	0.852 454 7	0.852 440	0.852 454 696 506
0.6	0.740 595 9	0.740 589	0.740 595 939 159
0.4	0.647 766 6	0.647 765	0.647 766 585 747
0.2	0.568 490 3	0.568 490	0.568 490 321 623
0.0	1/2	1/2	1/2
-0.2	0.440 813 2	0.440 813	0.440 813 209 205
-0.4	0.390 058 0	0.390 004	0.390 057 988 652
-0.6	0.346 967 9	0.346 814	0.346 967 773 049
-0.8	0.310 643 9	0.310 444	0.310 643 770 245
-1.0	0.280 085 9	0.279 890	0.280 085 758 710

self-adjoint property of the integral operator, it becomes evident from Eq. (13) that the largest eigenvalue ρ satisfies the inequality,

$$\rho \geq \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \int_0^{\infty} f(x) K(x,y) f(y) dx dy. \quad (14)$$

One can then use any trial function $f(x)$ containing one or more variational parameters and then maximize the right-hand side of Eq. (14) with respect to these parameters to derive a rigorous lower bound for $\rho(a)$ for arbitrary $0 < a < 1$.

The limiting forms of the true eigenfunction $g(x)$ in Eq. (13) for $a \rightarrow 0$ and $a \rightarrow 1$ can be easily worked out, and suggest a trial function of the form $f(x) = A(b+x) \exp(-\lambda x^2/2)$. The amplitude A is fixed by the normalization condition, $\int_0^{\infty} f^2(x) dx = 1$, while b and λ are the two variational parameters. The right-hand side of the inequality in Eq. (14) can then be evaluated in closed form and the optimization with respect to b and λ performed. The resulting variational estimate turns out to be very accurate for all $0 < a < 1$, when compared to numerical results, and agrees with the perturbative results to at least four or five decimal places.

Numerical integration. It is not difficult to integrate Eq. (7) directly. However, since $Q_n(x) \rightarrow 1$ as $x \rightarrow \infty$, numerically it is convenient to first make the transformation $Q_n(x) = G_n(x) \exp[(1-a^2)x^2/4]$ in Eq. (7) and then study the resulting equation for $G_n(x)$ by numerical iteration, with an arbitrary initial condition. For large n , $G_n(x)$ converges to $\rho^n g(x)$, where $g(x)$ is the solution of Eq. (13). The eigenvalue ρ is determined from the slope of the log-linear plot of $A_n = \int_0^{\infty} G_n(x) dx \sim \rho^n$ versus n . In Table I, we compare the numerical, variational, and perturbative estimates of ρ . The differences are small in all cases, and the variational bound is satisfied.

The eigenfunction $q(x)$ of Eq. (8) can also be calculated by using the series (9), with the coefficients $\{b_n\}$ obtained from the corresponding eigenvector of the matrix A , Eq. (12). It is shown, for $a = 0.5$, as the lower curve in Fig. 1.

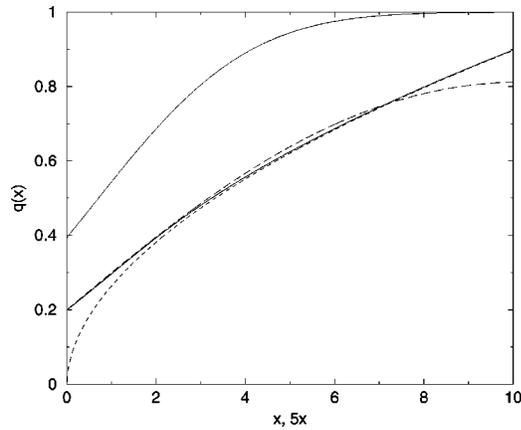


FIG. 1. The eigenfunctions $q(x)$ for $a=0.5$ (lower curve) and $a=2.0$ (upper curve, abscissa $=5x$). Solid lines—perturbative results; long-dashed—variational; dashed—asymptotic result $q(x) \sim x^\nu$, with $\nu = \ln \rho / \ln a \approx 0.530661$ for $a=0.5$.

The asymptotic large- x behavior (dashed curve) can be obtained analytically by noting that for large x we can set the lower limit in Eq. (8) to minus infinity with negligible error. The resulting equation can be solved exactly [12], with solution $q(x) = \exp(X^2/4)D_\nu(X)$, where $X = (1 - a^2)^{1/2}x$, $D_\nu(X)$ is the parabolic cylinder function, and $\nu = \ln \rho / \ln a$. The asymptotic behavior is $q(x) \sim x^\nu$. The variational trial function, however, misses this asymptotic behavior (see Fig. 1) even though the variational eigenvalue is very accurate.

Although Eq. (7) was derived for $a \geq 0$, one can also study this equation or, equivalently, Eqs. (8) and (13), for negative a . Is there a physical meaning for negative a ? Let $R_n(x)$ denote discrete “alternating” persistence, being the probability that, starting at $x > 0$ (x is related to X as before) at $T=0$, the particle’s position changes sign at alternate discrete points up to the n th step. Then $R_n(x)$ evolves via the recurrence equation,

$$R_{n+1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \exp[-(y-ax)^2/2] R_n(y) dy. \quad (15)$$

Changing $y \rightarrow -y$ inside the integral, and using $R_n(y) = R_n(-y)$ (since the process has zero mean), we find Eq. (15) reduces to Eq. (7) with a replaced by $-a$. Thus, $R_n(x, a) = Q_n(x, -a)$ and hence, the largest eigenvalue $\rho(a)$ for negative a governs the asymptotic decay of “alternating” discrete persistence. We also note that while, for $a > 0$, $Q_n(x) \sim \rho^n(a)q(x)$ for large n only for $a < 1$ [for $a > 1$, $Q_n(x)$ approaches a steady state, as will be seen later on], for negative a , $Q_n(x) \sim \rho^n(a)q(x)$ for all $a < 0$. Furthermore, from Eq. (12), one has the symmetry relation $\rho(1/a) = |a|\rho(a)$, which can be used to obtain ρ (and the corresponding eigenfunction) for $a < -1$ from the results for $-1 < a \leq 0$. In particular, $\rho \rightarrow 1/2$ for $a \rightarrow 0$ implies $\rho \rightarrow 1/2|a|$ for $a \rightarrow -\infty$.

Finally, we turn to the unstable potential, $\mu < 0$, i.e., $a = e^{-\mu\Delta T} > 1$. As in the continuous case, we expect that the solution of Eq. (7) for $a > 1$ will reach a steady state for large n , $Q_n(x) \rightarrow q(x)$, where $q(x)$ will satisfy Eq. (8), but with $\rho = 1$. Evidently $q(x)$ will depend on a , and in the limit $a \rightarrow 1^+$ (i.e., $\Delta T \rightarrow 0$) it reduces to the continuous result obtained from Eq. (4). For general $a > 1$, it is again possible to obtain accurate variational and very accurate perturbative estimates for $q(x)$. We omit the details here since they are somewhat similar to the $a < 1$ case. In Fig. 1, we plot the perturbative $q(x)$ for $a=2$ (upper curve). The variational result, and the numerical result obtained from direct iteration of Eq. (7), are both indistinguishable from the plotted curve. Note that the case $a < -1$, discussed in the preceding paragraph, corresponds to alternating persistence in an unstable potential, which does *not* approach a steady state.

In summary, we have shown that the discrete persistence due to the finite size of the time windows differs considerably from the continuous persistence usually studied, and we have computed explicitly this nontrivial effect analytically for a simple Markov model. The work extending some of the techniques developed here to more realistic non-Markov processes is in progress. We conclude by noting the recent examples of discrete time persistence in dynamical systems [13].

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