

## Dynamics of efficiency: A simple model

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We propose a simple model that describes the dynamics of efficiencies of competing agents. Agents communicate leading to increase of efficiencies of underachievers, and an efficiency of each agent can increase or decrease irrespectively of other agents. When the rate of deleterious changes exceeds a certain threshold, the system falls into a stagnant phase. In the opposite situation, the average efficiency improves with asymptotically constant rate and the efficiency distribution has a finite width. The leading algebraic corrections to the asymptotic growth rate are also computed.

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Nonequilibrium statistical mechanics applies to diverse fields from biology to finance, social and cognitive sciences [1–5]. Indeed, the framework of nonequilibrium statistical mechanics is ideally suited for describing systems composed of many units that interact according to simple rules and exhibit a complex large-scale behavior. Thus, the important task is to construct simple stochastic models incorporating basic characteristics of the dynamics of systems under study, which can then be analyzed by employing existing tools of nonequilibrium statistical mechanics. The hope is that these models can reproduce essential features of the original systems and enhance understanding of the dynamics of these systems.

In this Rapid Communication, we introduce a model where competing units interact with each other. Despite its simplicity, the model exhibits rich phenomenology, including a nonequilibrium depinning phase transition. In addition, the model is analytically tractable by using the techniques developed in the context of front propagation problems [7,8]. The model can be considered as a polynuclear growth model with desorption where the degrees of freedom are the heights of a growing interface, a language more familiar to the statistical physics community [6]. However, the rules of the model are also suited to an economic situation where the degrees of freedom are the efficiencies of competing agents. Throughout the rest of the paper, we shall use the language of efficiency, though we stress that the focus or the results need not be limited only to the economic situation.

Our model mimics the dynamics of efficiencies of competing agents which could be airlines, travel agencies, insurance companies, etc. In today's competing global economy, the performance of a company is continuously judged in the market and the index of performance depends on how efficient the company is. Rather than trying to incorporate all details of performances of competing agents, we choose a model that accounts for the dynamics of efficiency in the simplest form. We represent the efficiency of each agent by a single nonnegative number. The efficiency of every agent can, independent of other agents, increase or decrease stochastically by a certain amount which we set equal to unity. In addition, the agents interact with each other which is the

fundamental driving mechanism for economy. We assume that the interaction equates the efficiencies of underachievers to the efficiencies of better performing agents.

The efficiency model formalizing the above features is defined as follows. Let  $h_i(t)$  be the efficiency of agent  $i$  at time  $t$ . Efficiencies  $h_i$ 's are non-negative integer numbers which evolve stochastically. Specifically, in an infinitesimal time interval  $\Delta t$ , every  $h_i(t)$  can change as follows:

(i)  $h_i(t) \rightarrow \max[h_i(t), h_j(t)]$  with probability  $\Delta t$ , where the agent  $j$  is chosen randomly. This move is due to the fact that each agent tries to equal its efficiency to that of a better performing agent in order to stay competitive.

(ii)  $h_i(t) \rightarrow h_i(t) + 1$  with probability  $p\Delta t$ . This incorporates the fact that each agent can increase its efficiency, say due to innovations, irrespective of other agents.

(iii)  $h_i(t) \rightarrow h_i(t) - 1$  with probability  $q\theta(h_i(t))\Delta t$ , where  $\theta(x)$  is the Heaviside step function. This corresponds to the fact that each agent can lose its efficiency due to unforeseen problems such as labor strikes. Note, however, that since  $h_i(t) \geq 0$ , this move can occur only when  $h_i(t) \geq 1$ .

(iv) With probability  $1 - [1 + p + q\theta(h_i(t))]\Delta t$ , the efficiency  $h_i(t)$  remains unchanged.

The model exhibits rich behavior, including a delocalization phase transition as the parameters  $p$  and  $q$  are varied. Above a critical line,  $p > p_c(q)$ , the average efficiency increases linearly with time; for  $p \leq p_c(q)$ , the system is stagnant, i.e., the efficiency distribution becomes stationary in the large time limit. This delocalization (or depinning) phase transition is dynamical in nature and is different from the depinning transitions found in equilibrium systems with quenched disorder. Similar delocalization transitions have recently been found in a variety of nonequilibrium processes [9–13].

Let  $P(h, t)$  denote the fraction of agents with efficiency  $h$  at time  $t$ . One can easily write the evolution equation for  $P(h, t)$  by counting all possible gain and loss terms. For  $h \geq 1$ , this equation reads

$$\begin{aligned} \frac{dP(h,t)}{dt} = & -P(h,t) \sum_{h'=h+1}^{\infty} P(h',t) - (p+q)P(h,t) \\ & + qP(h+1,t) + pP(h-1,t) \\ & + P(h,t) \sum_{h'=0}^{h-1} P(h',t). \end{aligned} \quad (1)$$

In writing Eq. (1), we have taken into account that when the total number of agents diverges the joint probability distribution  $P(h,h',t)$  of two agents having efficiencies  $h$  and  $h'$  factorizes,  $P(h,h',t)=P(h,t)P(h',t)$ , and the mean-field theory becomes exact.

It proves convenient to consider the cumulative distribution,  $F(h,t)=\sum_{h' \geq h} P(h',t)$ . From Eq. (1), we immediately derive the evolution equation for  $F(h,t)$ ,

$$\begin{aligned} \frac{dF(h,t)}{dt} = & -F^2(h,t) + (1-p-q)F(h,t) + qF(h+1,t) \\ & + pF(h-1,t). \end{aligned} \quad (2)$$

Note that this equation is valid for all  $h \geq 1$ , and by the probability sum rule we have  $F(0,t)=1$  for arbitrary  $t$ . Also,  $F(h,t) \rightarrow 0$  as  $h \rightarrow \infty$  for all  $t$ .

Equation (2) is a nonlinear difference-differential equation and is, in general, hard to solve exactly. Fortunately, many asymptotic properties can be derived analytically without solving Eq. (2). First we note that  $F(h,t)$  approaches a traveling wave form as it follows, e.g., from direct numerical integration of Eq. (2). Thus, we seek a solution of the form  $F(h,t)=f(h-vt)$ . By inserting it into Eq. (2) we find that  $f(x)$  satisfies

$$\begin{aligned} -v \frac{df(x)}{dx} = & -f^2(x) + (1-p-q)f(x) + qf(x+1) \\ & + pf(x-1), \end{aligned} \quad (3)$$

which should be solved subject to the boundary conditions  $f(-\infty)=1$  and  $f(\infty)=0$ . To determine  $v$ , we linearize Eq. (3) in the tail region,  $x \rightarrow \infty$ . The resulting linear equation admits an exponential solution,  $f(x) \sim \exp(-\lambda x)$ . By inserting this asymptotics into the linearized version of Eq. (3) we find that the growth rate  $v(\lambda)$  is related to the decay exponent  $\lambda$  via

$$v(\lambda) = \frac{1-p-q+qe^{-\lambda}+pe^{\lambda}}{\lambda}. \quad (4)$$

Thus, we have a family of eigenvalues parametrized by  $\lambda$ . According to a general selection principle that applies to a wide class of nonlinear equations [7,8], only one specific rate out of this family of possible  $v$ 's is selected. Usually, the minimum rate is selected. For sufficiently steep initial conditions, the minimum rate is universal, while extended initial conditions might affect the magnitude of the *admissible* minimum rate.

The function  $v(\lambda)$  in Eq. (4) has a unique minimum at  $\lambda=\lambda^*$  given by the solution of  $dv/d\lambda=0$ , or

$$1-p-q+qe^{-\lambda^*}+pe^{\lambda^*}=\lambda^*[-qe^{-\lambda^*}+pe^{\lambda^*}]. \quad (5)$$

The corresponding minimum rate  $v_{\min}(p,q) \equiv v(\lambda^*)$  is given by Eq. (4), or  $v_{\min}=-qe^{-\lambda^*}+pe^{\lambda^*}$  as it follows from Eq. (5). An analysis of Eqs. (4) and (5) shows that there exists a critical line  $p_c(q)$  in the  $(p,q)$  plane,

$$p_c(q)=\begin{cases} 1+q-2\sqrt{q} & \text{for } q \geq 1 \\ 0 & \text{for } q \leq 1, \end{cases} \quad (6)$$

such that  $v(\lambda) > 0$  for all  $\lambda \geq 0$  as long as  $p > p_c(q)$ . For a fixed  $q$ , as  $p \rightarrow p_c(q)$  from above,  $v_{\min} \rightarrow 0$  and for  $0 < p < p_c(q)$ , the curve  $v(\lambda)$  crosses zero at  $\lambda=\lambda_1$  and  $\lambda=\lambda_2$  with  $\lambda_2 > \lambda_1$ . When  $\lambda_1 < \lambda < \lambda_2$ ,  $v(\lambda)$  becomes negative. This tells us that there might be no traveling wave solution for  $0 < p < p_c(q)$ , and we anticipate that the efficiency distribution should become stationary. Note that for  $q < 1$ ,  $p_c(q)=0$  and this regime does not occur.

With the above picture in mind, we now discuss the selection principle more carefully. Consider an exponentially decaying initial condition,  $F(h,0) \sim e^{-\alpha h}$  with  $\alpha > 0$ . When  $p > p_c(q)$ , the rate is positive for all  $\lambda > 0$  and  $v(\lambda)$  has a unique minimum at  $\lambda=\lambda^*$ . Applying the selection principle we find that for sufficiently steep initial conditions,  $\alpha > \lambda^*$ , the selected growth rate is  $v_{\min}=v(\lambda^*)$ . Consider now sufficiently extended initial conditions,  $\alpha < \lambda^*$ . In this case,  $f(x)$  must decay at most as  $e^{-\alpha x}$  and therefore the growth rate is selected among  $v(\lambda)$ , Eq. (4), with  $\lambda \leq \alpha$ . The selection principle now implies that the selected rate is  $v=v(\alpha)$ .

When  $p \leq p_c(q)$ , we must separately consider two cases:  $q > 1$  and  $q \leq 1$ . For  $q > 1$ ,  $v(\lambda)$  as given by Eq. (4) becomes negative in the region  $\lambda_1 < \lambda < \lambda_2$ . We find that for all  $\alpha < \lambda_1$ , the system still admits a traveling wave solution and the selected rate is  $v=v(\alpha)$ . However, for  $\alpha > \lambda_1$ , the system no longer admits a traveling wave solution. Instead, the distribution  $F(h,t)$  reaches a stationary limit  $P_\infty(h)$  as  $t \rightarrow \infty$ . By putting the time derivative equal to zero on the left-hand side of Eq. (2), we find that the stationary efficiency distribution decays exponentially,  $F_\infty(h) \sim e^{-\mu h}$ , with

$$\mu(p,q)=\ln\left[\frac{-1+p+q+\sqrt{(1-p-q)^2-4pq}}{2p}\right]. \quad (7)$$

Note that  $\mu(p,q)$  is real below the critical line, i.e., when  $q > 1$  and  $p \leq p_c(q)$ . Interestingly,  $\mu(p,q)$  remains finite on the critical line  $p=p_c(q)$ . From Eqs. (7) and (6) we find the critical decay rate for  $q > 1$ :

$$\mu_c(q) \equiv \mu[p_c(q),q] = -\ln[1-q^{-1/2}]. \quad (8)$$

For  $q \leq 1$ ,  $p_c(q)=0$ . When  $p \rightarrow 0$ , we have  $v_{\min} \rightarrow 0$  and  $\lambda^* \rightarrow \infty$ . The divergence of the decay exponent  $\lambda^*$  indicates that when  $p=0$  and  $q < 1$ , the system still admits a traveling wave solution and the selected rate is  $v=v(\alpha)$  if we start with an exponentially decaying initial condition,  $F(h,0) \sim e^{-\alpha h}$ . Of course, for compact initial conditions [i.e., when  $F(h,0)=0$  for sufficiently large  $h$ ], the efficiency dis-

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tribution becomes stationary in the long time limit. We have verified all the above assertions via direct numerical integration of Eq. (2).

Although one cannot provide *explicit* expressions for the growth rate in the developing phase, near the critical line  $p_c(q)$  the growth rate considerably simplifies. First we note that on general scaling grounds one would guess that above the critical line, the growth rate  $v_{\min}(p,q)$  should be a function of  $p - p_c$  with critical behavior

$$v_{\min} \sim (p - p_c)^\beta. \quad (9)$$

The actual behavior is found by a straightforward analysis of Eqs. (4) and (5) to yield

$$v_{\min} \rightarrow \begin{cases} \frac{p - p_c(q)}{(\sqrt{q} - 1)\mu_c(q)} & \text{for } q > 1 \\ 4 \frac{\sqrt{p}}{\ln(1/p)} & \text{for } q = 1 \\ \frac{1 - q}{\ln(1/p)} & \text{for } 0 \leq q < 1, \end{cases} \quad (10)$$

where  $\mu_c(q)$  is given by Eq. (7). Equation (10) implies that the mobility exponent  $\beta$  in the scaling relation (9) is equal to 1, 1/2, and 0 for  $q > 1$ ,  $q = 1$ , and  $q < 1$ , respectively. In the last situation ( $q < 1$  and  $p \rightarrow 0$ ), the growth rate still approaches to zero but it occurs in an extremely slow inverse logarithmic fashion.

The relaxation of the growth rate  $v(t)$  towards its asymptotic value  $v_{\min}$  exhibits an interesting algebraic behavior. Specifically, the leading correction is proportional to  $t^{-1}$ , the next is of order  $t^{-3/2}$ , etc. Similar  $t^{-1}$  correction was first derived by Bramson in the context of a reaction-diffusion equation [8], and was subsequently rederived and generalized by van Saarloos [14] and Brunet and Derrida [15]. The next correction was recently derived by Ubert and van Saarloos [16]. In contrast to Refs. [8,14–16], we consider the difference-differential equation. Fortunately, the techniques [8,14–16] still apply [13,17]. Following, for instance, an approach of Ref. [16], one finds

$$v(t) = v_{\min} - \frac{3}{2\lambda^*} t^{-1} + At^{-3/2} + O(t^{-2}), \quad (11)$$

with  $A = 3\pi^{1/2}\{2(qe^{-\lambda^*} + pe^{\lambda^*})\}^{-1/2}(\lambda^*)^{-2}$ . The explicitly displayed terms are *universal*; they do not depend on initial condition as long as it is steep enough (i.e., it falls off faster than  $e^{-\lambda^*x}$ ). The following terms in Eq. (11) starting from the  $O(t^{-2})$  correction are nonuniversal. Thus, not only any sufficiently steep initial profile relaxes to a unique profile, the approach to that profile occurs along a (asymptotically) unique trajectory. Note also that the very slow  $t^{-1}$  relaxation of the growth rate leads to a logarithmic correction to the average efficiency,

$$\langle h \rangle = v_{\min}t - \frac{3}{2\lambda^*} \ln t + O(1). \quad (12)$$

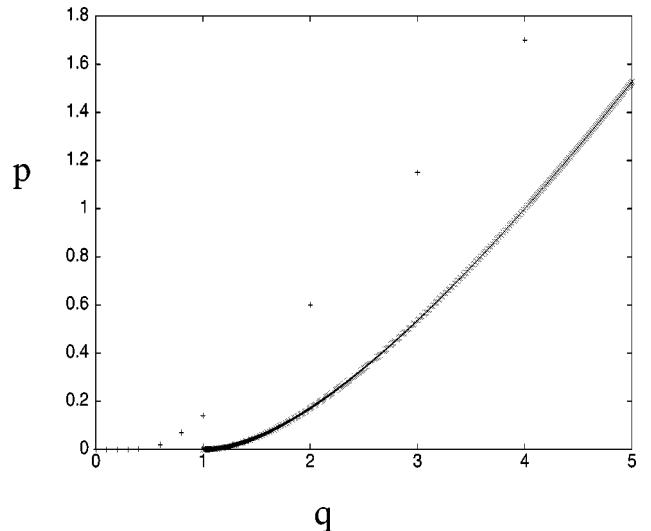


FIG. 1. Thick line represents the critical locus,  $p_c(q) = 1 + q - 2\sqrt{q}$ , for the mean-field theory. The '+'s indicate numerically obtained critical points in one dimension. For sharply decaying initial conditions, the system is developing when  $p > p_c(q)$  and stagnant when  $p \leq p_c(q)$ .

Thus, for  $p > p_c(q)$ , the system is in the developing phase, with  $\langle h \rangle$  increasing according to Eq. (12). For  $p \leq p_c(q)$  with  $q > 1$ , the system is localized and  $\langle h \rangle$  approaches a time-independent constant in the long time limit. For  $p = 0$  and  $q < 1$ , the system is in the developing phase for unbounded initial efficiency distributions, with the growth rate dependent on initial conditions. For economically relevant compact initial conditions, the regime  $p = 0$  and  $q < 1$  belongs to the stagnant phase.

The mean-field version of the efficiency model is natural in the interconnected modern economy. In a situation with limited communication, however, the efficiency model in a low-dimensional space rather than in the fully connected graph might be more appropriate. In this case, agents are placed on a finite-dimensional lattice. The microscopic dynamical steps (i)–(iv) remain the same except that in move (i), the agent  $j$  is chosen to be one of the nearest neighbors of  $i$ . Unlike the mean-field theory, the correlations between  $h_i$ 's at different sites remain nonzero in finite dimensions even in the thermodynamic limit. We have studied this model numerically in one dimension. The results are shown for lattice size  $L = 1000$  (we verified that for such large systems, the finite size effect is insignificant). Once again, there is a delocalization transition in the  $(p,q)$  plane across a critical line, as shown in Fig. 1. The efficiency distribution  $P(h,t)$  at different times in both phases is presented in Fig. 2.

We now stress important differences between mean-field and finite-dimensional situations. In the former case the nature of the two phases depends on the steepness parameter  $\alpha$ , while in one dimension the nature of the final state is independent of  $\alpha$ . For example, in the developing phase the system always has a traveling wave solution with a rate that depends on  $p$  and  $q$  but does not depend on  $\alpha$ . We have tested this fact numerically for several values of  $\alpha$ . This result is rather counterintuitive, as it suggests that correla-

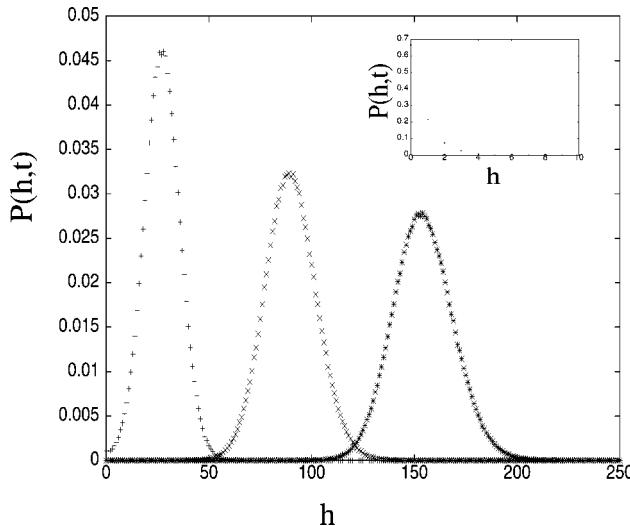


FIG. 2. Distribution  $P(h,t)$  at times  $t=200, 600$ , and  $1000$  in the moving phase for  $p=3, q=4$ . The inset shows the distribution at the same times in the localized phase for  $p=1, q=4$ . For  $q=4$ , the critical point is  $p_c(q)\approx 1.7$ .

tions seem to restore universality that mean-field theory lacks. Another important distinction is a very different behavior of the width of the efficiency distribution in the developing phase. Indeed, in mean field the width is constant, while in one dimension it increases with time (see Fig. 2). Moreover, the width increases as a power law,  $w=\sqrt{\langle h^2 \rangle - \langle h \rangle^2} \sim t^\beta$  for large  $t$ , with  $\beta\approx 0.31$  in  $(1+1)$  dimensions.

In  $(d+1)$  dimensions, one can interpret the efficiency  $h_i(t)$  as the height of a surface growing on a  $d$ -dimensional substrate. In this language, our efficiency model represents a continuous time polynuclear growth (PNG) model with desorption. The continuous PNG model without desorption has been studied within mean-field theory [18] and was found to be always in the moving phase as expected. From the general analogy to PNG models, we expect that the moving phase in the efficiency model corresponds to a growing interface belonging to the Kardar-Parisi-Zhang (KPZ) universality class [6]. The numerically obtained width exponent  $\beta\approx 0.31$  in  $(1+1)$  dimensions is consistent with the KPZ prediction  $\beta=1/3$ . It would be interesting to determine the universality class of the delocalization transition. Phase transitions in several PNG models in  $(1+1)$  dimensions belong to the directed percolation (DP) universality class (see, e.g., Ref. [19]). Other similar growth models exhibit phase transitions that do not belong to the DP universality class [9,11]. It remains an open question whether the phase transition in the efficiency model in  $(1+1)$  dimensions belongs to the DP universality class.

In summary, we have investigated a simple model of the dynamics of efficiencies of competing agents. The model takes into account stochastic increase and decrease of the efficiency of every agent, independent of other agents, and interaction between the agents which equates the efficiencies of underachievers to that of better performing agents. We have shown that the model displays a depinning transition from a stagnant to a growing phase.

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