Hitting probability for anomalous diffusion processes

Satya N. Majumdar and Alberto Rosso
CNRS - Université Paris-Sud, LPTMS, UMR8626 - Bât. 100, 91405 Orsay Cedex, France

Andrea Zoia
CEA/Saclay, DEN/DM2S/SERMA/LTSD, Bât. 454, 91191 Gif-sur-Yvette Cedex, France

We present the universal features of the hitting probability $Q(x,L)$, the probability that a generic stochastic process starting at $x$ and evolving in a box $[0,L]$ hits the upper boundary $L$ before hitting the lower boundary at 0. For a generic self-affine process (describing, for instance, the polymer translocation through a nanopore) we show that $Q(x,L) = Q(x/L)$ and the scaling function $Q(z) \sim z^\phi$ as $z \to 0$ with $\phi = \theta/H$ where $H$ and $\theta$ are respectively the Hurst exponent and the persistence exponent of the process. This result is verified in several exact calculations including when the process represents the position of a particle diffusing in a disordered potential. We also provide numerical supports for our analytical results.

The transfer of DNA, RNA and proteins through cell membranes is key to understanding several biological processes [1]. The transport of polymer molecules across nanopores is also relevant in many chemical and industrial applications [2]. A fundamental question concerns whether a polymer, once penetrated into the pore, will eventually complete its transit. The answer is naturally whether a polymer, once penetrated into the pore, will eventually complete its transit. The answer is naturally whether a polymer, once penetrated into the pore, will eventually complete its transit. The answer is naturally whether a polymer, once penetrated into the pore, will eventually complete its transit. The answer is naturally whether a polymer, once penetrated into the pore, will eventually complete its transit.

It is useful to summarize our main results which are threefold: i) For self-affine processes, we show that generically $Q(z) \sim z^\phi$ for small $z$, where $\phi = \theta/H$, and $\theta$ is the so-called persistence exponent [12] of the same process in a semi-infinite geometry. ii) For a particle diffusing in a disordered potential $V(X)$, we provide an exact formula for $Q(x,L)$ valid for arbitrary $V(X)$ which incidentally also allows us to compute the persistence exponent of a particle diffusing in a self-affine disordered potential. iii) The function $Q(z)$ is explicitly known for some anomalous diffusion processes. Amazingly, we find that these apparently different-looking formulae can be cast in the same super-universal form, when expressed in terms of the exponent $\phi$. This naturally raises the question: how generic is this super-universality? We provide numerical evidences that indeed in some cases the super-

FIG. 1: Left. The evolution of a stochastic process initiated at $X(0) = x$ and terminated upon exiting from the box of size $L$. Right. The function $Q(z)$ as given by Eq. (8) for different values of the exponent $\phi$. 
universal validity is violated and we discuss its limit of validity.

**Self-affine processes.** To compute \( Q(x, L) \) in a box geometry, it is useful first to relate it to another quantity associated with the same process \( X(t) \), but now in a semi-infinite geometry \([0, \infty)\]. Consider a process \( X(t) \) in \([0, \infty)\), starting at \( x \) and absorbed at the origin for the first time at \( t_f \) — the first-passage time. Let \( m \) denote the maximum of this process till \( t_f \) (see Fig. 2 left). Then, it is clear that \( 1 - Q(x, L) \), the probability that the particle exits the box through the origin (and not through \( L \)), is precisely equal to the probability that the maximum \( m \) of the process in \([0, \infty)\) till \( t_f \) stays below \( L \), i.e., the cumulative distribution of \( m \), \( \text{Prob}[m \leq L|x] \), in the semi-infinite geometry. The distribution of \( m \) is, in turn, related to the distribution of the first-passage time \( t_f \). Let \( q(x, T) = \text{Prob}[t_f \geq T|x] \) denote the cumulative probability of \( t_f \), which is also the survival probability of the particle starting at \( x \) in the semi-infinite geometry. One knows that for generic self-affine processes \( q(x, T) = q(x/T^H) \). For large \( T \), \( q(x, T) \sim T^{-\theta} \), where \( \theta \) is the persistence exponent of the process \([12]\). This implies the scaling function \( y(\theta) \sim y^{\theta/H} \) for small \( y \) \([7]\). Noting that \( m \sim t_f^H \) for self-affine processes, it follows that \( Q(x, L) = 1 - \text{Prob}[m \leq L|x] \approx \text{Prob}[m \geq L|x] \approx \text{Prob}[t_f \geq L^{1/H}|x] = q(x/L) \). This demonstrates the scaling behavior anticipated before, namely, \( Q(x, L) = Q(x/L) \), where \( Q(z) = q(z) \). Moreover, since \( q(y) \sim y^{\theta/H} \) for small \( y \), we get \( Q(z) \sim z^\phi \) for small \( z \), with \( \phi = \theta/H \). For example, for Brownian motion \( H = 1/2 \) and \( \theta = 1/2 \), hence \( \phi = 1 \), in accordance with the exact result \( Q(z) = z \). For the subclass of self-affine processes with stationary increments, the same exponent \( \phi \) happens to describe the vanishing of the probability density close to an absorbing boundary \([7]\).

Our general prediction \( Q(z) \sim z^\phi \) for small \( z \) can be verified explicitly for some self-affine processes where \( Q(z) \) can be computed exactly, as discussed later. Moreover, we have numerically verified that this conjecture holds also for the fractional Brownian motion (fBm), i.e., a self-affine Gaussian process defined by the following autocorrelation function

\[
\langle X(t_1)X(t_2) \rangle = \frac{1}{2} \left( t_1^{2H} + t_2^{2H} - |t_1 - t_2|^{2H} \right),
\]

with \( 0 < H < 1 \) \([13, 14]\). In \([7]\), we have proposed fBm as a natural candidate for describing the time evolution of the translocation coordinate. For this process, the persistence exponent is known, \( \theta = 1 - H \) \([14]\), so that \( \phi = (1 - H)/H \). An expedient algorithm for generating fBm paths is provided in \([15]\). The probability \( Q(z) \) can be numerically computed as follows. Given a realization of the process starting from the origin, we record its minimum and maximum values for increasing time; the process is halted when \( X_{\text{max}} - X_{\text{min}} \geq L \). If the last updated quantity is \( X_{\text{min}} \), the contribution to \( Q(x, L) \) is 0 for \( x \in (0, L - X_{\text{max}}) \) and 1 for \( x \in (L - X_{\text{max}}, L) \). In the opposite case, the contribution is 0 for \( x \in (0, -X_{\text{min}}) \) and 1 otherwise. All simulations are performed by averaging over \( 10^6 \) samples. Fig. 3 shows the agreement between numerical simulations and predicted scaling of \( Q(z) \) for different values of the parameter \( H \).

**Disordered potential.** We next consider the stochastic motion of a single particle diffusing in a potential \( V(X) \), starting at the initial position \( x \). The dynamics is governed by the Langevin equation \( \dot{X}(t) = f[X(t)] + \eta(t) \), where \( X(0) = x \) and \( f(X) = -dV(X)/dX \) is the force and \( \eta(t) \) is a Gaussian white noise with \( \langle \eta(t) \eta(t') \rangle = \delta(t - t') \). To compute \( Q(x, L) \), the first step is to write a differential equation satisfied by \( Q(x, L) \), taking the initial position \( x \) as a variable and keeping the box size \( L \) fixed. Let us consider a small time interval \([0, \Delta t]\) at the beginning of the process. In this time interval, the particle moves from its initial position \( x \) to a new position...
the density of the particle to be at which can be interpreted as the equilibrium probability
potential-free particle, i.e., when the potential determine averaged hitting probability. An example where we can
related noise), yields an ordinary differential equation, for fBm processes: 
For fBm processes: \( H = 2/3 \) (\( \phi = 1/2 \)), with box size \( L = 200 \), and 
\( H = 3/4 \) (\( \phi = 1/3 \)), with box size \( L = 300 \). For fBm disordered potentials: \( H_v = 2/3 \) (\( \phi = 1/3 \)), with box size 
\( L = 10^4 \). For comparison, we display also the Sinai model 
\( H_v = 1/2 \) (\( \phi = 1/2 \)), with box size \( L = 10^4 \).

\[ x + \Delta x \text{ at time } \Delta t, \text{ where } \Delta x = f(x) \Delta t + \eta(0) \Delta t, \text{ being the noise variable that kicks in at time 0. Since the} \]

process is Markovian, the subsequent evolution does not know about the interval \([0, \Delta t]\), hence one gets

\[ Q(x, L) = \langle Q(x + f(x) \Delta t + \eta(0) \Delta t, L) \rangle, \]

(2)

where \( \langle \rangle \) denotes the average over the initial noise \( \eta(0) \). Expanding the rhs of Eq. (2) as a Taylor series in powers of \( \Delta t \), using \( \eta(0) = 0 \) and \( \eta^2(0) = 1/2 \Delta t \) (delta correlated noise), yields an ordinary differential equation,

\[ \frac{1}{2} Q''(x) + f(x) Q'(x) = 0. \]

Solving with boundary conditions \( Q(0, L) = 0 \) and \( Q(L, L) = 1 \) gives the exact result

\[ Q(x, L) = \frac{\int_0^x e^{2V(x')} dx'}{\int_0^L e^{2V(x')} dx'}. \]

(3)

valid for arbitrary potential \( V(X) \). Note that for a potential-free particle, i.e., \( V(X) = 0 \) in Eq. \( 3 \), we recover the Brownian result, \( Q(x, L) = x/L \).

Taking derivative with respect to \( x \) gives

\[ p_{eq}(x, L) = \frac{\partial}{\partial x} Q(x, L) = \frac{e^{2V(x)}}{\int_0^L e^{2V(x')} dx'}. \]

(4)

which can be interpreted as the equilibrium probability density of the particle to be at \( x \) in presence of a potential \( -V(X) \). When \( V(X) \) is a realization of a disordered potential, it is natural to introduce \( Q(X, L) \), the disordered-averaged hitting probability. An example where we can determine \( Q(X, L) \) explicitly is the classical Sinai model, i.e., when the potential \( V(X) \) is a trajectory of a Brownian motion in space, \( V(X) \sim X^{1/2} \) \([16]\). For this model the \( p_{eq}(x, L) \) can be computed exactly \([17, 18]\).

\[ p_{eq}(x, L) = \frac{1}{\pi} \frac{1}{\sqrt{x(L-x)}}. \]

(5)

Thus, \( Q(x, L) = Q(z = x/L) \) again satisfies the generic scaling with a scaling function form with

\[ Q(z) = \frac{2}{\pi} \arcsin (\sqrt{z}). \]

(6)

Note that close to the origin \( Q(z) \sim z^\phi \) with \( \phi = 1/2 \). On the other hand, it is well known that in the Sinai potential the particle evolves very slowly with time, \( X \sim \ln^2(t) \), showing a self-affine scaling in the variable \( T = \log t \), with \( H = 2 \). For this model, it is also known that the survival probability decays as \( 1/ \log t \), i.e., \( T^{-3} \), with \( \theta = 1 \) \([19, 20]\). Thus, \( \theta/H = 1/2 = \phi \), in accordance with our general scaling prediction.

We next consider a generic self-affine potential, \( V(X) \sim X^{HV} \) (with \( V(0) = 0 \)), the Sinai potential being a special case with \( H_v = 1/2 \). We show that \( p_{eq}(x, L) \) for such a potential is related to the probability density of the location \( x_m \) of the maximum of the potential \( V(X) \) over \( X \in [0, L] \). We rewrite Eq. \( 4 \) as \( p_{eq}(x, L) = [\int_0^1 e^{2V(x')} - V'(x') dx']^{-1} \), rescale variables \( x' \to x/L \) and \( x \to xL \) and use the self-affine property \( V(xL) = L^{HV} V(x) \) to obtain \( p_{eq}(x, L) = [\int_0^1 e^{2HV} V'(x') - V(x') dx']^{-1} \), for large \( L \), using a steepest decent method, we immediately see that, for each realization of the disorder potential \( V(X) \), \( p_{eq}(x, L) \simeq \delta(x-x_m) \) where \( x_m \) denotes the position where \( V(X) \) is maximum. This observation has two immediate consequences: (i) By integrating over \( x \), we get, for each realization, \( Q(x, \alpha) \simeq \theta(x-x_m) \). This means that, for any given realization, if the starting position \( x \) is to the left (right) of the location \( x_m \) of the maximum, \( Q(x, \alpha) \simeq 0 \) (respectively \( Q(x, \alpha) \to 1 \)) indicating that the particle exits the box through 0 (through \( L \)) as depicted in Fig. 2 right. (ii) By taking average over the disorder, we get

\[ p_{eq}(x, L) \simeq p_m(x, L) \]

(7)

where \( p_m(x, L) \) is the probability density that the maximum of the potential \( V(X) \) over \([0, L]\) is located at \( x \). For example, for the Sinai case, one knows from Lévy’s arcsine law \([8]\) that \( p_m(x, L) = 1/\pi \sqrt{L-x} \). Thus, in this case the relation (7) is verified by the exact result (5). However, the relation (7) is more general and holds for arbitrary self-affine potential. We remark here that physically the relation (7) reflects the fact that in a self-affine potential \( -V(X) \) in which the particle is at equilibrium, the limit \( L \to \infty \) limit is equivalent to the zero temperature \( T \to 0 \) limit forcing the particle to the minimum of the potential \( -V(X) \) or equivalently to the maximum \( x_m \) of \( V(X) \).

A useful consequence of (7) is that it allows us to relate the persistence or the survival probability of a particle moving in a disordered self-affine potential to the statistical properties of the potential \( V(X) \) itself. The disordered potential \( V(X) \sim X^{HV} \) (we assume \( V(0) = 0 \) can itself be regarded as a stochastic process with the
space coordinate $X$ playing the role of ‘time’. So, the probability that $V(X)$ stays below (or above) the level $X = 0$ up to a distance $L$ decays, for large $L$, as $L^{-\theta V}$ where $\theta V$ is the spatial persistence exponent \cite{zolotarev} of $V(X)$. For example, for the Sinai potential (brownian motion in space), $\theta V = 1/2$. The pair of exponents $(H V, \theta V)$ associated with the potential can now be related to the corresponding exponents associated with the temporal motion of the particle in this potential. By Arrhenius’ law for the activated dynamics, the time required for particles corresponding exponents associated with the temporal motion can now be related to the correlation functions. In \cite{Zapotoczny-Krakow2012}, the persistence exponent is $\theta = 1/4$, so that $\theta = \theta V / H V = 1/6$. Bicout and Burkhardt \cite{Bicout1998} also computed the full exit probability $Q(x, L)$. One can again recast this formula in the same super-universal form \cite{Zapotoczny-Krakow2012} with $\phi = 1/6$.

Based on these special cases, it may be tempting to conjecture that the full function $Q(z)$ for arbitrary anomalous diffusion processes has the super-universal form \cite{Zapotoczny-Krakow2012}, the only information about the process enters this formula through a single exponent $\phi$. However, this turns out not to be the case, and we are able to show notable counterexamples. In Fig. 4 we compute the hitting probability for fBm self-affine processes and for particles diffusing in fBm disordered potentials, and display the numerical difference with respect to formula \cite{Zapotoczny-Krakow2012}, with the appropriate exponent $\phi$. We find that in neither case $Q(z)$ can be described by the super-universal form \cite{Zapotoczny-Krakow2012}.

\cite{Zapotoczny-Krakow2012} A. Zoia, A. Rosso, S.N. Majumdar, Phys. Rev. Lett. 102, 120602 (2009).
\cite{Zapotoczny-Krakow2012} T. Dieker, http://www2.isye.gatech.edu/~adieker3/.
\cite{Zapotoczny-Krakow2012} P. Le Doussal, D.S. Fisher, C. Monthus, Phys. Rev. E