Erratum


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The paper [1] deals with the study of N non-intersecting Brownian motions on a line segment \([0, L]\) with three different types of boundary conditions: (I) periodic, (II) absorbing and (III) reflecting boundary conditions. We showed in particular that the normalized reunion probabilities of these Brownian motions in the three models can be mapped to the partition function of two-dimensional continuum Yang–Mills theory on a sphere respectively with gauge groups \(U(N), Sp(2N)\) and \(SO(2N)\). Consequently, we showed that in each of these Brownian motion models, as one varies the system size \(L\), a third order phase transition occurs at a critical value \(L = L_c(N) \sim \sqrt{N}\) in the large \(N\) limit. Close to the critical point, the reunion probability, properly centered and scaled, can be expressed in terms of Tracy–Widom distributions describing the probability distribution of the largest eigenvalue of random matrices. While our statements for case (I) and case (II), the later being the most interesting one, are perfectly correct, the results for the rescaled reunion probability \(\tilde{E}_N(L)\) in the limit of large \(N\) for the case (III), which we only discussed briefly in Ref. [1], is actually erroneous: this is the reason for this erratum. We wrote indeed in Ref. [1] that \(\tilde{E}_N(L)\) converges, when \(N \to \infty\), to the Tracy–Widom distribution corresponding to GOE random matrices. We correct this wrong statement and find instead that \(\tilde{E}_N(L)\) converges, when \(N \to \infty\), to the ratio of \(F_2(t)/F_1(t)\) where \(F_2(t)\) and \(F_1(t)\) are respectively the Tracy–Widom distributions corresponding respectively to GUE and GOE random matrices, in terms of the variable \(t = 2^{11/6}|L - \sqrt{2N}|\).

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Before arriving at this result, we first correct several typos. First we give the correct expression for the probability distribution of the longest path in the Hammersley model given in Eq. (10):

\[
\lim_{N \to \infty} \Pr\left( \frac{h - N}{(N/2)^{1/3}} < t \right) = \exp\left( -\int_{t}^{\infty} (s-t)q^2(s) \, ds \right),
\]

(1)

while the scaling variable given in Ref. [1] is \((h - N/2)/((N/2)^{1/3})\).

We then correct the error of sign which is in Eq. (12). The correct expression of the Tracy–Widom distribution for GOE, \(F_1(t)\) is actually

\[
F_1(t) := \exp\left( -\frac{1}{2} \int_{t}^{\infty} ((s-t)q^2(s) + q(s)) \, ds \right),
\]

where \(q(t)\) is the Hastings–McLeod solution of the Painlevé II differential equation

\[
q''(t) = 2q^3(t) + tq(t), \quad q(t) \sim \text{Ai}(t).
\]

(2)

We also remind, for later purpose, the expression of the Tracy–Widom distribution for GOE, \(F_2(t)\) [given correctly in Eq. (11) of Ref. [1]]

\[
F_2(t) := \exp\left( -\int_{t}^{\infty} (s-t)q^2(s) \, ds \right).
\]

Before turning to the model III, we also want to correct yet another error of sign in Eq. (89), which should indeed be [given Eq. (88) together with the relation between \(f_1(x)\) and \(u(t) = q(t)\) given in Eq. (57)]

\[
\frac{d^2}{dt^2} \log \tilde{F}_N(\sqrt{2N(1 + t/(2^{7/3}N^{2/3}))}) = -\frac{1}{2}(q^2(t) - q'(t)),
\]

(3)

which yields indeed

\[
\lim_{N \to \infty} \tilde{F}_N(\sqrt{2N(1 + t/(2^{7/3}N^{2/3}))}) = F_1(t),
\]

(4)

in agreement with Eq. (2).

We now turn to this model III, where we consider a model of non-intersecting Brownian motions where the walkers move again on a finite line segment \([0, L]\), but this time with reflecting boundary conditions at both boundaries 0 and \(L\). The walkers start in the vicinity of the origin at time \(\tau = 0\) and we consider the reunion probability \(R_{III}^L(1)\) that they reunite at time \(\tau = 1\) at the origin. Following models I and II as introduced in Eqs. (13) and (16) of Ref. [1], we define the normalized reunion probability

\[
\tilde{E}_N(L) = \frac{R_{III}^L(1)}{R_{III}^\infty(1)},
\]

(5)

that is independent of the starting positions \(\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}\) in the limit when all the \(\epsilon_i\)'s tend to zero and hence depends only on \(N\) and \(L\). We want to correct the two typos which were in present in formula (19) and (20) in Ref. [1] which should be actually:

\[
\tilde{E}_N(L) = \frac{C_N}{L^{2N^2-N}} \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} \Delta^2(n_1^2, \ldots, n_N^2) e^{-(\pi^2/(2L^2)) \sum_{j=1}^{N} n_j^2},
\]

(6)
and the prefactor
\[ C_N = \frac{\pi^{2N^2-N^2} 2^{N/2-N^2}}{\prod_{j=0}^{N-1} \Gamma(2+j)\Gamma(1/2+j)}, \]
ensures that \( \tilde{E}_N(L \to \infty) = 1 \). Note that the formula (6) differs from Eq. (19) in Ref. [1] by a factor of 2 in the argument of the exponential, while the present formula (7) carries the correct power of 2, at variance with the formula in Eq. (20) given in Ref. [1].

It is then perfectly correct to say that this ratio of reunion probability (6) is related to the partition function of two-dimensional Yang–Mills theory on the sphere with gauge group SO(2N):
\[ \tilde{E}_N(L) \propto Z\left(A = \frac{2\pi^2 N}{L^2}; \text{SO}(2N)\right), \]
where \( Z(A; \text{SO}(2N)) \) is the partition function of 2d Yang–Mills theory on the sphere, with gauge group \( \text{SO}(2N) \) computed with the heat kernel action, as explained in detail in Ref. [1].

This identification (8) allows to conclude that \( \tilde{E}_N(L) \) exhibits a third order phase transition (of Douglas–Kazakov type [2]) when \( L \) crosses the critical value \( L_c = \sqrt{2N} \). Around this critical value, the behavior of \( \tilde{E}_N(L) \) is described by a double scaling limit, which is indeed similar to the one presented in Section 4.2 of Ref. [1], where we studied in detail the model (II) with absorbing boundary conditions. One finds, similarly to the case (II), that in this scaling regime \( \tilde{E}_N(L) \) is a function of the scaling variable \( t = 2^{11/6} |L - \sqrt{2N}| N^{1/6} \) but the associated scaling function turns out to be different from \( F_1(t) \), the Tracy–Widom distribution for the Gaussian Orthogonal Ensemble (GOE) as we claimed in Ref. [1]. Indeed, we find instead
\[ \tilde{E}_N(L) \to \frac{F_2(t)}{F_1(t)}, \quad t = 2^{11/6} |L - \sqrt{2N}| N^{1/6}, \]
where \( F_2(t) \) is the Tracy–Widom distribution for the Gaussian Unitary Ensemble. This result (9) is obtained by performing a calculation which is similar to the one presented for the model (II), which corresponds to YM theory with gauge group \( \text{Sp}(2N) \), yielding to Eq. (3) above. However, for the present model (6) one finds
\[ \frac{d^2}{dt^2} \log \tilde{E}_N(\sqrt{2N}(1 + t/(2^{7/3} N^{2/3}))) = -\frac{1}{2} (q^2(t) + q'(t)). \]

Note in particular the different sign of the coefficient in front of \( q'(t) \), compared to Eq. (3). This different sign, which we actually missed in Ref. [1] yields actually to a different limiting function, given by
\[ \lim_{N \to \infty} \tilde{E}_N(\sqrt{2N}(1 + t/(2^{7/3} N^{2/3}))) = \exp\left(-\frac{1}{2} \int_{t}^{\infty} ((s-t)q^2(s) - q(s)) \, ds \right) = \frac{F_2(t)}{F_1(t)}, \]
where the result on the last line is straightforwardly obtained from Eqs. (2) and (3) above.

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References