# Growth of Long-Range Correlations after a Quench in Conserved-Order-Parameter Systems

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We develop a general Langevin formalism for the dynamics after a quench to a critical point or an ordered phase, and use this to study a few specific cases. We present a general argument that for $d$ spatial dimensions and conserved order parameter, the local autocorrelations decay as $\langle\phi(\mathbf{r},0)\,\phi(\mathbf{r},t)\rangle \sim L^{-d}(t)$, where $L(t)$ is the correlation length at time $t$, and $\phi$ is the order parameter. We also present new analytical and numerical results for the coarsening process after a quench to zero temperature in the ferromagnetic Ising chain with conserved magnetization.

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The kinetics of phase separation following a quench from a high temperature disordered phase to a low temperature ordered phase has been a subject of continual and growing interest [1]. After the quench, domains of the equilibrium ordered phases form and coarsen with time as the system achieves local equilibrium on larger and larger length scales. This coarsening process generally exhibits dynamic scaling at the late stage of growth, i.e., the system is described by a single length scale $L(t)$ (namely the characteristic linear size of the domains) which grows with time as $L(t) \sim t^n$. According to the dynamic scaling hypothesis [2], the equal time correlation function, $G(r,t) = \langle\phi(\mathbf{r}',t)\,\phi(\mathbf{r}'+\mathbf{r},t)\rangle$ [where $\phi(\mathbf{r},t)$ denotes the order parameter field] scales as $G(\mathbf{r},t) \sim f[r/L(t)]$ at large times after the quench. The average ($\langle\,\rangle$) is over all possible initial conditions and the histories of evolution. Another quantity of interest is the local autocorrelation function, $A(t) = \langle\phi(\mathbf{r},0)\,\phi(\mathbf{r},t)\rangle$, which measures the correlation with the initial condition ($t=0$) and decays as $L^{-\lambda}(t)$. For quenches to the critical point $T = T_c$, the correlation length grows with exponent $n_c = 1/z$, where $z$ is the usual dynamic critical exponent. The exponent $\lambda_c$ for the autocorrelation, $A(t) \sim L^{-\lambda_c} \sim t^{-\lambda_c/z}$, after a quench to $T_c$ is a new nonequilibrium critical exponent [3,4].

The exponents $n$, $\lambda$, $\lambda_c$ and the scaling functions depend crucially on the conservation laws satisfied by the dynamics. For example, the exponent $n$ has been argued to be $1/2$ and $1/3$ for nonconserved [5] and conserved [6] scalar order parameter, respectively, for dimensions $d \geq 2$. For vector order parameter, they are respectively $1/2$ and $1/4$ for $d > 2$ [7]. In $d = 1$, the exact solution of the nonconserved zero temperature Glauber dynamics gives $n = 1/2$ and $\lambda = \lambda_c = 1$ [8]. A few results are known for $\lambda$ or $\lambda_c$ in higher dimensions. For the nonconserved case, they have been obtained analytically for the $O(m)$ model in the limit $m \to \infty$ for $d > 2$ [4,9]. For 2D nonconserved Ising ($m = 1$) dynamics, $\lambda = 5/4$ theoretically [10] and experimentally [11]. Apart from these, there have been numerical studies for $\lambda$ [12] and $\lambda_c$ [3] in higher $d$ for the nonconserved case. Comparatively, not much is known about $\lambda$ or $\lambda_c$ in the conserved case. For the conserved $O(m)$ model in the $m \to \infty$ limit, Bray has shown $\lambda_c = d$ for all $d$ [13]. Fisher and Huse [10] have proposed bounds on $\lambda$, namely, $d/2 \leq \lambda \leq d$. The autocorrelations after a quench have been measured using video microscopy [11] and may also be measured via the autocorrelations of the speckle pattern seen in coherent light or x-ray scattering.

In this Letter, we present a simple general argument, assuming scaling, that concludes $\lambda = \lambda_c = d$ for the conserved case for all $d$. This result applies for both scalar and vector order parameters. In particular, we study, both analytically and numerically, the 1D conserved dynamics for the Ising model in the limit $T \to 0$ and show that $\lambda = \lambda_c = 1$ at par with our general result.

Let us first present the general argument as follows. Let $\{\phi(\mathbf{k},t)\}$ denote the Fourier transforms of the order parameter field $\phi(\mathbf{r},t)$. Then the evolution equation of $\phi(\mathbf{k},t)$, assuming a noisy relaxational dynamics, can be written quite generally as

$$\frac{\partial\phi(\mathbf{k},t)}{\partial t} = D(\mathbf{k},\{\phi(\mathbf{k}',t)\}) + \eta(\mathbf{k},t,\{\phi(\mathbf{k}',t)\}), \quad (1)$$

where $D$ and $\eta$ denote, respectively, the deterministic and the stochastic part of the evolution. Thus for fixed $\mathbf{k}$ and $\{\phi(\mathbf{k}',t)\}$, the average value of $\partial\phi(\mathbf{k},t)/\partial t$ is $D(\mathbf{k},\{\phi(\mathbf{k}',t)\})$ while the average value of $\eta$ vanishes. For continuum Langevin-type models with nonconserved (model A) and conserved (model B) order parameters [14], the noise $\eta$ is generally assumed to be uncorrelated Gaussian white noise with zero average and an amplitude independent of $\{\phi(\mathbf{k}',t)\}$ and $t$. The mean square amplitude is $\mathbf{k}$ independent for model A and is proportional to $k^2$ for model B. But more generally, and specifically for Glauber (spin-flip) and Kawasaki (spin-exchange) dynamics of discrete hard spin systems, the noise amplitude may depend on both $\mathbf{k}$ and $\{\phi(\mathbf{k}',t)\}$. It is then easy to see that the two-time correlation function $C(\mathbf{k},0,t) = \langle\phi(-\mathbf{k},0)\phi(\mathbf{k},t)\rangle$ evolves as $C(\mathbf{k},0,t) = C(\mathbf{k},0,0)\exp[\int_0^t \Gamma(\mathbf{k},t')\,dt']$ where

$$\Gamma(\mathbf{k},t) = \langle\phi(-\mathbf{k},0)\,D(\mathbf{k},\{\phi(\mathbf{k}',t)\})\rangle / \langle\phi(-\mathbf{k},0)\,\phi(\mathbf{k},t)\rangle\,.$$

$C(\mathbf{k},0,0)$, for random initial conditions, is a constant of $O(1)$ independent of $\mathbf{k}$. Assuming $C(\mathbf{k},0,t)$ scales, it



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follows that $\Gamma(\mathbf{k},t)$ should scale as $\Gamma(\mathbf{k},t) \sim (1/t)\,\gamma[kL(t)]$ for large $t$, where $\gamma(x)$ is a scaling function. Assuming $L(t) \sim t^n$, in the large $t$ scaling limit we get $C(\mathbf{k},0,t) \sim \exp[n^{-1}\int_{kL_0}^{kL(t)} \gamma(y)\,dy/y]$, where $L_0$ is a microscopic length, representing the initial correlation length. Writing $\gamma(y) = \gamma(0) + \gamma_1(y)$, where $\gamma_1(y) \to 0$ as $y \to 0$, one gets

$$C(\mathbf{k},0,t) \sim [L(t)]^{\gamma(0)/n} F_1[kL(t)] \qquad (2)$$

where $F_1(x) = \exp[(1/n)\int_0^x dy\,\gamma_1(y)/y]$. Clearly then, the local autocorrelation function $A(t) \sim \int C(\mathbf{k},0,t)\,d^d\mathbf{k} \sim [L(t)]^{-\lambda}$, with $\lambda = d - \gamma(0)/n$. In the conserved case where the total order parameter, $\phi(\mathbf{k}=\mathbf{0})$, is constant, $C(\mathbf{0},0,t)$ is independent of $t$ implying $\Gamma(\mathbf{0},t) = 0$, and hence $\gamma(0) = 0$ identically. Thus the scaling assumption implies $\lambda = d$ for the conserved case quite generally. This general argument applies even for quenches to $T_c$ and therefore we expect $\lambda_c = d$ as well. Recent numerical simulation [15] of the critical dynamics of a spin-exchange kinetic Ising model in 2D yields $\lambda_c \approx 2.0$, in excellent agreement with our general result. For the nonconserved case, on the other hand, $\gamma(0)$ may be nonzero and $\lambda$ is thus different from $d$ in general.

As a specific example of this general result, we now consider the 1D nearest neighbor spin-exchange (Kawasaki) kinetic Ising model. The 1D conserved dynamics is special and interesting for another reason. In 1D, as opposed to higher dimensions, the ordered phases coexist only at $T = 0$ and therefore coarsening can occur only at $T = 0$. For the nonconserved Glauber dynamics at $T = 0$ in 1D, domains grow indefinitely via the random diffusion and annihilation of kinks (domain walls). However, in the conserved case at $T = 0$, the domains stop growing indefinitely when the system gets "trapped" in a metastable excited state with isolated domain walls [16]. Thus the question is what happens to the coarsening process in 1D for the conserved case.

Coarsening occurs only in the limit $T \to 0$ and on a special time scale. For finite $T$, the 1D Kawasaki dynamics has been studied numerically [17, 18]. By considering the most elementary thermal excitation out of a metastable state, it was argued in [18] that for small $T$, each domain as a whole performs random walk with a rate proportional to $1/L$ where $L$ is the length of the domain. It is then easy to see that if one rescales time by the factor $\exp(-4J/k_BT)$ (where $J$ is the exchange coupling and $k_B$ is Boltzmann's constant) and takes the limit $T \to 0$, then on this rescaled time scale $\tau = t\exp(-4J/k_BT)$, the original Kawasaki dynamics is effectively described by the above "domain" model of Cornell, Kaski, and Stinchecombe [18]. Coarsening occurs on this rescaled time scale via the merging of the diffusing domains and it is then easy to argue that the length scale of domains grow as $L(\tau) \sim \tau^{1/3}$ as in higher dimensions. However, it is important to realize that this scaling holds only on the rescaled time scale $\tau$ with the $T \to 0$ limit taken properly. Thus to obtain the real scaling behavior of the coarsening process, it is necessary to study and simulate directly the stochastic domain model as described above rather than the finite temperature Kawasaki dynamics as studied previously [17, 18]. We study this domain model both numerically and analytically.

Let us first discuss the results of direct simulation of the domain model. The simulation picks a domain $i$ with probability $L_i^{-1}/\sum_j L_j^{-1}$ and then randomly chooses to move it to the left or right. It then increments the time $\tau$ by a random number taken from the Poisson distribution $p(x) = \mu\exp(-\mu x)$ with $\mu = \sum_j L_j^{-1}$. This process is quite rapid and we have been able to simulate very large times of order $10^8$. The simulation is typically done on a lattice with $10^6$ sites but was repeated for smaller lattices to verify that there are no finite size effects in the time regime we study. The dynamics for $L = 1, 2$ is not modeled precisely in our simulation, but this should not affect universal properties at late times.

In Fig. 1, we show the average domain size $\langle L\rangle$ (averaged over 9 runs) as a function of time on a log-log plot. We get a very good straight line at late times with a slope $0.33 \pm 0.01$ supporting $n = 1/3$. In Fig. 2, we plot the equal time correlation function $G(r,t)$ averaged over 350 runs as a function of the scaled distance $r/\langle L\rangle$ at five different times. These curves fall on top of each other exhibiting dynamic scaling at late times. Shown in Fig. 3 is the two-time correlation function $\langle s(r',0)\,s(r'+r,t)\rangle$ [normalized by the autocorrelation function $A(t) \equiv \langle s(0,0)s(0,t)\rangle$] as a function of the scaled distance for five different times. In Fig. 4, we show the autocorrelation function $A(t)$ as a function of $L(t)$ on a log-log plot. The slope of the straight line gives $\lambda = 1.00 \pm 0.01$.

We now show explicitly that the 1D dynamics both for nonconserved and conserved cases can be cast in the form of Eq. (1). Consider an arbitrary spin configuration

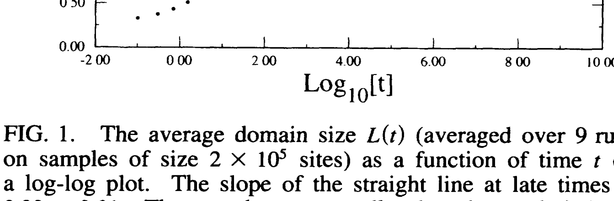


FIG. 1. The average domain size $L(t)$ (averaged over 9 runs on samples of size $2 \times 10^5$ sites) as a function of time $t$ on a log-log plot. The slope of the straight line at late times is $0.33 \pm 0.01$. The error bars are smaller than the symbol sizes.

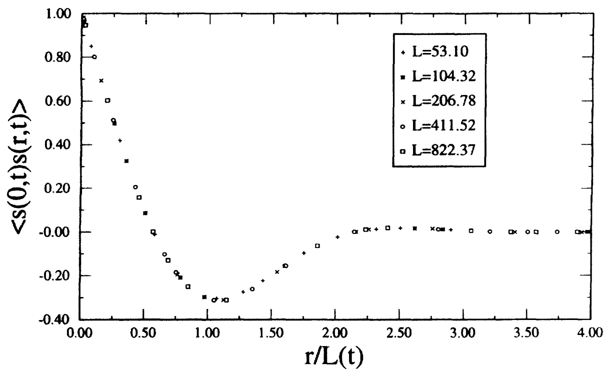


FIG. 2. The equal-time correlation function $\langle s(0,t)\, s(r,t)\rangle$ (averaged over 350 runs on samples of size $10^6$ sites) as a function of the scaled distance $r/L(t)$ at five different times (shown by symbols) when $L$ is 53.10, 104.32, 206.78, 411.52, and 822.37.

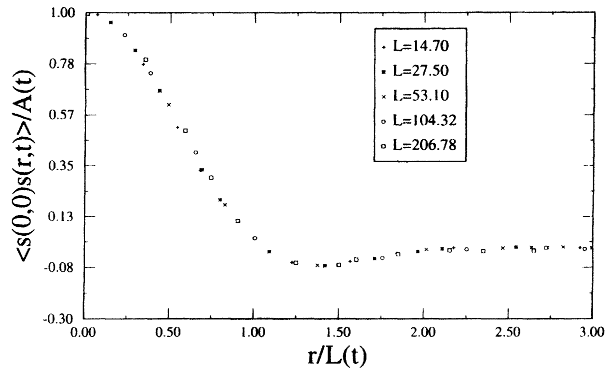


FIG. 3. The two-time correlation function $\langle s(0,0)\, s(r,t)\rangle$ [normalized by the autocorrelation $A(t)$ and averaged over 350 runs as in Fig. 2] as a function of the scaled distance $r/L(t)$ when $L$ is 14.70, 27.50, 53.10, 104.32, and 206.78.

$\{\phi(x,t);\ \phi(x,t) = \pm 1\}$ on a 1D lattice. The Fourier transform $\phi(k,t) = (1/\sqrt{N}) \sum_x \phi(x,t) \exp(ikx)$, where $N$ is the total number of spins, can be written in a domain wall representation as

$$\phi(k,t) = \frac{1}{i\sqrt{N}\,\sin(k/2)} \sum_n (-1)^n \exp(ikx_n)\,, \qquad (3)$$

where the domains are sequenced as $1, 2, \ldots, n, \ldots$ and $x_n$ represents the position of the domain wall separating the $(n-1)$th domain and the $n$th domain. Let us first consider the nonconserved zero temperature Glauber dynamics. In this case, each wall executes a simple random walk. The increment in $\phi(k,t)$ in a small time interval $\Delta t$ is given by

$$\Delta\phi(k,t) = \frac{1}{i\sqrt{N}\,\sin(k/2)} \sum_n (-1)^n \exp(ikx_n)[\exp(ik\zeta_n) - 1]\,, \qquad (4)$$

where the $\zeta_n(t)$'s are the distances moved by the walls in $\Delta t$. They are independent random variables each taking values $+1$, $-1$, or 0 with probabilities $\frac{1}{2}\Delta t$, $\frac{1}{2}\Delta t$, and $1 - \Delta t$, respectively. Taking the limit $\Delta t \to 0$, the equation of motion can then be written in the form of Eq. (1) with $D(k,t) = -(1 - \cos k)\phi(k,t)$ and the stochastic part $\eta(k,t)$ is Gaussian with zero average and $\langle \eta(k',t')\, \eta(k,t)\rangle = 4\cos^2(k/2)\rho(t)\delta_{k+k'}\delta(t-t')$, where $\rho(t) = L^{-1}(t)$ is the average density of domain walls at time $t$. Thus, following our general discussion, $\Gamma(k,t) = -(1 - \cos k)$ and in the scaling limit of large $t$ and small $k$, $\Gamma(k,t) \sim (1/t)\gamma[kL(t)]$ with $L(t) \sim t^{1/2}$ and $\gamma(x) = x^2$. Thus $\gamma(0) = 0$ and hence $\lambda = 1$ for the nonconserved case. The structure factor $S(k,t) = \langle \phi(-k,t)\, \phi(k,t)\rangle$ can also be computed exactly by this method and scales as $S(k,t) \sim L(t)F[kL(t)]$ with $F(x) = x^{-1} \int_0^x \exp[-(x^2 - y^2)/2]\,dy$. In fact, to our knowledge, this is a new way of solving the zero-temperature 1D Glauber dynamics.

We now turn to the conserved case. Proceeding as above, the increment in $\phi(k,t)$ in time $\Delta t$ is

$$\Delta\phi(k,t) = \frac{1}{2i\sqrt{N}\,\sin(k/2)} \sum_D (-1)^{n_D}[\exp(ikx_D^L) - \exp(ikx_D^R)][\exp(ik\zeta_D) - 1]\,, \qquad (5)$$

where $n_D$ denotes the sequence number of the domain $D$ and $x_D^L$ and $x_D^R$ denote the locations of the left and right wall of the domain $D$. The sum is over all domains. The random variables $\zeta_D$'s are once again independent and each takes values $+1$ with probability $(1/2L_D)\Delta t$, $-1$ with probability $(1/2L_D)\Delta t$, and 0 with probability $1 - (1/L_D)\Delta t$, where $L_D$ is the length of the domain. Once again, the equation of motion can be broken into two parts as in Eq. (1) with the deterministic part $D(k,t)$ given by

$$D(k,t) = -\frac{1 - \cos k}{2i\sqrt{N}\,\sin(k/2)} \sum_D \frac{(-1)^{n_D}}{L_D} [\exp(ikx_D^L) - \exp(ikx_D^R)] \qquad (6)$$

and the noise $\eta(k,t)$ has zero average. Assuming further that $\rho(L_D,t)$, the fraction of domains of length $L_D$ at time $t$ scales as $\rho(L,t) \sim L(t)^{-1} f_0(L_D/L(t))$, the two point correlator of the noise can be shown to be

$$\langle \eta(k',t')\eta(k,t)\rangle = L^{-2}(t) g[kL(t)]\, \delta_{k+k'}\delta(t-t')\,, \qquad (7)$$

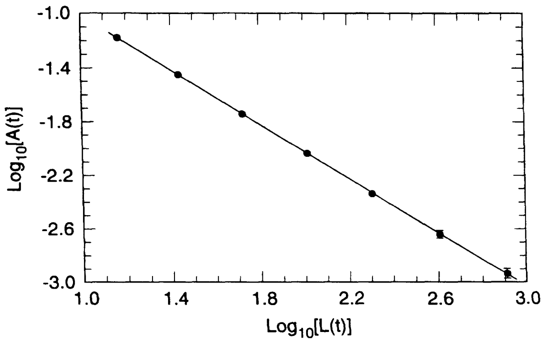


FIG. 4. The autocorrelation function $A(t) = \langle s(r,0)\, s(r,t)\rangle$ (averaged over 350 runs as in Fig. 3) as a function of $L(t)$ on a log-log plot. The slope is $-1.00 \pm 0.01$.

where $g(x) = 2\int_0^\infty dy\, f_0(y)\,(1 - \cos xy)/y$. Note that the deterministic part $D(k,t)$ is no longer simply proportional to $\phi(k,t)$ as it is in the Glauber case. Thus although we get an explicit form of $\Gamma(k,t)$, it is hard to express the scaling function $\gamma(x)$ in a closed form. However, it is clear that $\gamma(x) \to 0$ as $x \to 0$ yielding $\lambda = 1$ at par with the simulation and our general argument. For the same reason, we could not get a closed form expression of the structure factor scaling function $F(x)$. However, for small $x = kL(t)$, it can be seen that $F(x) = ax^2 + O(x^4)$ where the constant $a = -\frac{1}{2}\int_0^\infty f(y) y^2\, dy$, with $f(y)$ the scaling function for the equal time correlation in real space (as shown in Fig. 2); numerically we find $a = 0.168$. This behavior is in contrast to higher dimensions where simulations [19] and phenomenological arguments [20] show that $F(x) \sim x^\delta$ with $\delta \geq 4$ for quenches into the ordered phase. In one dimension, the presence of the quadratic term in the structure factor for small wave vector indicates that for small $k$, the growth of structure factor is solely due to the noise term which thus plays an important role in the coarsening process in 1D as opposed to higher dimensions where the noise has been argued to be irrelevant [2, 21].

Another interesting case where one can compute the function $\Gamma(k,t)$ explicitly is the $m \to \infty$ limit of the $O(m)$ model. In the nonconserved case [9], it can be simply shown that for large time $t$, $\Gamma(k,t) = (1/t)\gamma(k\sqrt{t})$ with $\gamma(x) = d/4 - x^2$. Thus $\gamma(0) = d/4$ and since $n = 1/2$, we get $\lambda = d - \gamma(0)/n = d/2$. In case of quench to the critical point [4], $\gamma(x) = \epsilon/4 - x^2$ for $2 < d \leq 4$, where $\epsilon = 4 - d$ and, therefore, $\lambda_c = d - \epsilon/2$. For $d \geq 4$, $\gamma(x) = -x^2$ and hence $\lambda_c = d$. For quenches to below $T_c$ in the conserved case, it is known [9] that scaling breaks down in the $m \to \infty$ limit due to the fact that the $t \to \infty$ and $m \to \infty$ limits do not commute [22]. However, for a conserved quench to the critical point, scaling is recovered [13] and then it can be shown that $\Gamma(k,t) = (1/t)\gamma(kt^{1/4})$ with $\gamma(x) = cx^2 - x^4$ where $c$ is a constant. Thus, $\gamma(0) = 0$ and hence $\lambda_c = d$ as expected.

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