Exact Dynamics of a Class of Aggregation Models

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The dynamics of a class of aggregation models proposed by Takayasu and co-workers is solved exactly in one dimension and in the mean field limit. These models describe the aggregation of positive and negative charges. In one dimension, we find the dynamical cluster-size exponents $z=\frac{3}{2}$ and $z_c=\frac{3}{4}$ when the average flux of injected charges is nonzero and zero, respectively. We also find the crossover exponent near the transition to be $\phi=\frac{4}{3}$. Within mean field theory, we find these exponents to be $z=2$, $z_c=1$, and $\phi=1$. Assuming dynamic scaling, we show that in any dimension, these exponents are related to one single static exponent.

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The study of irreversible processes evolving to a nonequilibrium steady state has a long history in statistical physics. Among these, systems automatically selecting steady states characterized by power law distributions have attracted much attention recently under the name of self-organized criticality [1,2]. A possible example of the class of self-organized systems is one involving aggregation. Aggregation is a typical irreversible process where diffusing particles coalesce on encounter, and has been studied over many years since the seminal work of Smoluchowski [3]. A wide class of aggregation systems are known to exhibit power law distributions such as the distribution of cluster sizes at the sol-gel transition point [4], and the pore size in aerosols [5]. Recently, Takayasu and co-workers [6] have proposed an aggregation model where constant injection of particles drives the system asymptotically into a steady state with power law distribution of particle masses, and thus exhibits self-organized criticality, as there is no fine tuning parameter in the model. The (1+1)-dimensional space-time geometry of this model was shown [7] to be identical to the geometry of the drainage area in Scheidegger's river model [8]. The space-time geometry of this model is also equivalent to the directed Abelian sandpile model [9].

A generalized version of this model was proposed by Takayasu [10] where the dynamical variables are charges rather than masses and can have both positive and negative values. The charge is conserved under aggregation and the injection process consists of raining charges of both signs steadily with a certain distribution $P(I)$ of the injected charge I. The steady state of this model in one dimension and in the mean field limit was solved exactly in [10] and was shown to have an interesting transition as a function of the average charge injected $\langle I\rangle$, at the symmetry point where $\langle I\rangle=0$. In one dimension, for $\langle I\rangle>0$, the distribution $p(s)$ of charge s in the steady state was shown to have an asymptotic one-sided power law tail $p(s)\sim s^{-4/3}$ for large positive s. For $\langle I\rangle=0$, the distribution is symmetric with a power law tail $p(s)\sim|s|^{-5/3}$.

However, for $\langle I\rangle>0$, the steady state distribution for negative s has not been studied in detail. Also, not much progress has been made in the study of the dynamics. Recently, Nagatini [11] has studied numerically the dynamics of the generalized Takayasu model in one dimension and suggested a dynamical scaling hypothesis for the positive charge distribution:

$$p(s,t)\sim s^{-\tau}f_+(s/t^z),\quad s,t\gg 1\,, \tag{1}$$

where $\tau=\frac{4}{3}$ is the exact static exponent and the dynamical cluster-size exponent $z=1.48\pm0.02$ for $\langle I\rangle>0$, and $\tau_c=\frac{5}{3}$ and $z_c=0.75\pm0.02$ for $\langle I\rangle=0$. He has called the transition a kinetic growth transition and has found that, near the transition point, the crossover time $\hat{t}$ scales as $\langle I\rangle^{-\phi}$, where the crossover exponent is 1.21 ± 0.02 numerically.

In this Letter, we first solve the dynamics of the generalized Takayasu model in one dimension and also in the mean field case exactly and show that the dynamical cluster-size exponents are, respectively, $z=\frac{3}{2}$ for $\langle I\rangle\neq0$, and $z_c=\frac{3}{4}$ for $\langle I\rangle=0$ in one dimension, and $z=2$ and $z_c=1$ in the mean field limit. Our exact calculation shows that the crossover exponent ϕ is $\frac{4}{3}$ in one dimension at variance with Nagatini's numerical value 1.21 ± 0.02. We believe that his method of extracting the crossover exponent may not have been accurate, and suggest a way of obtaining ϕ from the numerical data, preferable to the one used in [11]. In the mean field case, we find $\phi=1$. Finally, we prove quite generally that in any dimension, assuming dynamic scaling, there is only one independent exponent τ and all other exponents are related to τ via the scaling relations $z=1/(2-\tau)$, $\tau_c=2\tau-1$, $z_c=1/(4-2\tau)$, and $\phi=4-2\tau$. We then make a dynamic scaling hypothesis for $p(s,t)$ for large negative s near the critical point. This introduces a new length scale in the steady state and is consistent with the above results and also with Nagatini's simulation [11].

For simplicity, we describe the model in one dimension [6]. At each site i of a one-dimensional chain, there is at most one particle with a certain charge $S_i(n)$, at discrete time n. The dynamics is parallel. At each step, every particle either hops to its right with probability $\frac{1}{2}$ or

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stays back with probability $\frac{1}{2}$. If more than one particle happen to be at the same site, they coalesce to a single particle with charge equal to the sum of the individual charges. Then, a certain charge $I_i(n)$ is injected at each site i. Without loss of generality, we can assume that the average injected charge $\langle I\rangle \geq 0$. The dynamics is represented by the following stochastic equation:

$$S_i(n+1)=\sum_j w_{ij}S_j(n)+I_i(n)\,, \tag{2}$$

where w_{ij} is 0 or 1 with probability $\frac{1}{2}$ if $j=i$ or $j=i-1$, and $w_{ij}=0$ if $j\neq i$ and $j\neq i-1$. The distributions of the injected charge at each site are independent, identical, and time independent. The r-body characteristic function $Z(r,\rho,t)=\langle \exp[i\rho\sum_{j=1}^{r}S_j(t)]\rangle$, in the continuous time t, evolves from any given initial condition as [10]

$$\frac{\partial Z}{\partial t}(r,\rho,t)=\frac{\Phi(\rho)^r}{4}\{Z(r+1,\rho,t)+Z(r-1,\rho,t)+[2-4\Phi(\rho)^{-r}]Z(r,\rho,t)\}\,, \tag{3}$$

where $\Phi(\rho)=\langle e^{i\rho I}\rangle$, and with the boundary condition $Z(0,\rho,t)=1$. The large s behavior of $p(s,t)$ will be reflected in the small ρ behavior of

$$Z(1,\rho,t)=\int_{-\infty}^{+\infty}p(s,t)\exp(i\rho s)ds\,.$$

In fact, the dynamic scaling for $p(s,t)$ in Eq. (1) will determine the scaling behavior of $Z(1,\rho,t)$ for small ρ,

$$Z(1,\rho,t)-1\sim\rho^{\tau-1}F(t\rho^{1/z})\,. \tag{4}$$

The steady state can be analyzed exactly by equating the right-hand side (RHS) of Eq. (3) to zero. The behavior of $\hat{Z}(1,\rho)=Z(1,\rho,+\infty)$ for small ρ was shown to be [7,10]

$$\hat{Z}(1,\rho)-1\sim c_1\langle I\rangle^{1/3}\rho^{1/3}\ \text{for}\ \langle I\rangle\neq 0$$
$$\sim c_2\langle I^2\rangle^{1/3}|\rho|^{2/3}\ \text{for}\ \langle I\rangle=0\,, \tag{5}$$

which immediately gave $\tau=\frac{4}{3}$ and $\tau_c=\frac{5}{3}$.

We now proceed to solve the dynamics. First, we consider the case $\langle I\rangle=0$. We assume the distribution of I to have a finite second moment, and expand $\Phi(\rho)$ up to order ρ^2. We look for solutions of the form $Z(r,\rho,t)=\hat{Z}(r,\rho)+Q(r,\rho)\exp(-\lambda t)$, which gives from Eq. (3)

$$Q(r+1,\rho)+Q(r-1,\rho)-2[1-2\lambda+\rho^2\langle I^2\rangle r(1-\lambda)]Q(r,\rho)=0\,. \tag{6}$$

Thus, $Q(r,\rho)$ satisfies the recursion relations for the Bessel functions of both first and second kind [12]. However, since $Z(r,\rho,t)$ is bounded for large ρ, the only acceptable solution is the first kind $J_\nu(z)$. This gives

$$Q(r,\rho)=A_\lambda J_\nu\left[\frac{1}{\rho^2(1-\lambda)\langle I^2\rangle}\right],\quad \nu=\frac{1-2\lambda}{\rho^2(1-\lambda)\langle I^2\rangle}+r\,, \tag{7}$$

where the eigenvalues λ's are determined by inserting the boundary condition $Q(0,\rho)=0$ in Eq. (7). We note that a similar eigenvalue equation was obtained and analyzed by Racz [13] in studying the relaxation of homogeneous density fluctuations in a one-dimensional particles system with diffusion, annihilation, and steady input of particles. Following a similar analysis, we find that the smallest eigenvalue governing the long time decay is given by

$$\lambda_0=0.9279\langle I^2\rangle^{2/3}|\rho|^{4/3}\,, \tag{8}$$

where the constant in front of Eq. (8) is $2^{-4/3}|a_1|$, where a_1 is the first zero of the Airy function on the negative real axis. Thus, from (4) and (8), we immediately get $z_c=\frac{3}{4}$. For $\langle I\rangle\neq 0$, the analysis proceeds exactly the same way except that the leading term of $\Phi(\rho)$ is linear in ρ and the Bessel functions are of complex order and argument. However, the zeros of this complex function can still be found for large order and argument; the lowest eigenvalue scales as $\rho^{2/3}\langle I\rangle^{2/3}$, thus yielding $z=\frac{3}{2}$. The constants A_λ's in (7) are determined from the initial condition. This general solution thus establishes the scaling hypothesis (4) in one dimension.

To calculate the crossover exponent ϕ, we compute the moments of the distribution $p(s,t)$. Let us denote $\langle s_+\rangle(t)=\int_0^{+\infty}sp(s,t)ds$ and $\langle s_-\rangle(t)=\int_{-\infty}^{0}sp(s,t)ds$. Clearly, for $\langle I\rangle\neq 0$, $\langle s_+\rangle+\langle s_-\rangle=\langle s\rangle\sim\langle I\rangle t$, for large t. This follows from the fact that the total charge is conserved during the aggregation process. Also, $\langle s_+\rangle(t)-\langle s_-\rangle(t)=\langle|s|\rangle(t)$ can at most grow linearly in t. This is due to the fact that $\langle|s|\rangle(t)$ can never increase during the aggregation process. It can only increase through the injection at a steady rate $\langle|I|\rangle$. These two facts combined together give $\langle s_\pm\rangle(t)\sim A_\pm t$ where $A_++A_-=\langle I\rangle$ and $A_-\leq 0$. However, if A_- is nonzero, it would imply that the average size of the negative clusters would diverge in the same way as the average positive cluster size in the steady state. Physically, and as found numerically [11], $\langle s_-\rangle$ must be finite in the steady state as large negative clusters are statistically rare. Thus, $\langle s_-\rangle(t)$ cannot grow as fast as $\langle s_+\rangle(t)$, demanding that A_- be identically zero and $\langle s_+\rangle(t)\sim\langle s\rangle(t)\sim\langle I\rangle t$ for large t. We note that this argument is valid in any dimension. For $\langle I\rangle=0$, from dynamic scaling (1) (which we have established in one dimension), $\langle s_+\rangle(t)\sim t^{z_c(2-\tau_c)}\sim t^{1/4}$ for large t. This implies that $\langle s_+\rangle(t)\sim t^{1/4}g(\langle I\rangle t^{1/\phi})$ with $g(x)\sim x$ for large x and $g(x)\to$ const when $x\to 0$. This immediately gives $\phi=\frac{4}{3}$. In fact, for $\langle I\rangle\neq 0$, by taking successive derivatives of Eq. (3) with respect to ρ, and setting $\rho=0$, one can obtain the hierarchy of equations satisfied by the nth derivative $Z^{(n)}(r,0,t)$. The moments are then determined by the relation $\langle s^n\rangle(t)=(-i)^nZ^{(n)}(1,0,t)$. The continuous version (in r) of these equations can be shown to have a scaling solution. For instance, $Z^{(2)}(r,0,t)$ is found to be of the form $t^3G(rt^{-1/2})$ where $G(u)$ is related to the wave function of the sixth excited state of the harmonic oscillator and $G(u)\sim\langle I\rangle^2u$ at small u and $G(u)\sim\langle I\rangle^2u^2$ at large u. For $\langle I\rangle=0$, one can show directly from (3) that $\langle s^2\rangle(t)\sim t$. Thus, from the scaling of the second moment also, we find $\phi=\frac{4}{3}$. In [11], the crossover exponent was determined from the ratio of the second moment to $\langle s_+\rangle$, which from our analysis behaves like $t^{3/4}$ for $\langle I\rangle=0$, and $\langle I\rangle t^{3/2}$ otherwise, once again leading to the same value of ϕ. Although Nagatini's simulations agree well with our exact values $z=\frac{3}{4}$ and $z_c=\frac{3}{2}$, he obtained the value $\phi=1.21\pm0.02$, significantly smaller than our result. He determined the crossover time as the point at which the tangential line of slope 0.75 and 1.48 intersects, in a log-log plot of $\langle s_+^2\rangle/\langle s_+\rangle(t)$. Unfortunately, the scaling regime with slope 0.75 seems to be reached only for $|\langle I\rangle|<10^{-3}$, and the sampling time is too short for $|\langle I\rangle|>10^{-3}$, causing a systematic inaccuracy in ϕ. A more precise determination of ϕ can be done by directly analyzing the scaling of the coefficient of $t^{3/2}$ as a function of $\langle I\rangle$. We also find the general behavior of the nth moment to be $t^{z_c(n+1-\tau_c)}$ for $\langle I\rangle=0$, and $\langle I\rangle^n t^{z(n+1-\tau)}$ otherwise.

We next consider the mean field case where each particle can hop to any of N sites with probability $1/N$. In this case, the evolution equation for $Z(1,\rho,t)$, in the $N\to+\infty$ limit, can be directly shown to be

$$\frac{\partial Z}{\partial t}(1,\rho,t)=\Phi(\rho)\exp[Z(1,\rho,t)-1]-Z(1,\rho,t). \tag{9}$$

This equation can be easily analyzed in the limit $\rho\to0$. Writing $Z(1,\rho,t)=1-y(\rho,t)$, and noting that $y(\rho,t)$ is small for small ρ, we expand the exponential in (9) in powers of y. Keeping terms only up to order y^2, one easily finds that the steady state behavior is given by [10] $y(\rho,+\infty)\sim\langle I\rangle^{1/2}\rho^{1/2}$ for $\langle I\rangle\neq0$, and $y(\rho,+\infty)\sim\langle I^2\rangle^{1/2}\times|\rho|$ for $\langle I\rangle=0$, indicating $\tau=\frac{3}{2}$ and $\tau_c=2$. The dynamics can be analyzed once again by writing $y(\rho,t)=y(\rho,+\infty)+Q(\rho)\exp(-\lambda t)$, where λ is found to scale as $\langle I\rangle^{1/2}\rho^{1/2}$ for $\langle I\rangle\neq0$, and as $\langle I^2\rangle^{1/2}|\rho|$ for $\langle I\rangle=0$, which thus gives $z=2$ and $z_c=1$. We find that $\langle s^2\rangle=Z^{(2)}(1,0,t)$ scales as $\langle I\rangle^2t^3$ for $\langle I\rangle\neq0$, and as $\langle I^2\rangle t$ for $\langle I\rangle=0$, which immediately gives $\phi=1$. This is also consistent with the fact that $\langle s_+\rangle(t)\sim\langle I\rangle t$ for $\langle I\rangle\neq0$, and is bounded for $\langle I\rangle=0$.

In order to obtain the scaling relations in general dimension, we note that even for $d>1$, one can define an r-body generating function $Z(r,\rho,t)$, although, in general, it will depend on the shape of the r-body cluster. However, it is easy to verify that for independent, identical, and time-independent injection distributions, the dependence of $Z(r,\rho,t)$ on ρ is only via $\Phi(\rho)$. Therefore, any singular dependence of $Z(r,\rho,t)$ on ρ comes only through the singular dependence of $Z(r,\rho,t)$ on $\Phi(\rho)$. The difference in the exponents for $\langle I\rangle=0$ and $\langle I\rangle\neq0$ is a consequence of the fact that the leading term in $\Phi(\rho)$ is $\langle I\rangle i\rho$ in the first case, whereas it is $-\langle I^2\rangle\rho^2/2$ in the latter case. Assuming dynamic scaling in higher dimensions, we immediately get the scaling relations $\tau_c-1=2(\tau-1)$ and $z_c=z/2$. Note that if $\Phi(\rho)-1\sim\rho^\alpha$ with $\alpha\neq1$ and $\alpha\neq2$, which can arise if some moments of the distribution of I do not exist, the exponents will explicitly depend on α, and thus will not be universal. We do not study further this possibility. Furthermore, for $\langle I\rangle>0$, the average positive cluster size $\langle s_+\rangle(t)$ behaves like $\langle I\rangle t$ in any dimension, as already noticed. Comparing this with the scaling hypothesis (1), we immediately get the third scaling relation $(2-\tau)z=1$. Finally, noting that for $\langle I\rangle=0$, $\langle s_+\rangle(t)\sim t^{z_c(2-\tau_c)}$ from Eq. (1), and as stated above that $\langle s_+\rangle(t)\sim\langle I\rangle t$ for $\langle I\rangle>0$, we get the crossover exponent $\phi=1/[1-z_c(2-\tau_c)]$. These four scaling relations, combined together, give $z=1/(2-\tau)$, $\tau_c=2\tau-1$, $z_c=1/(4-2\tau)$, and $\phi=4-2\tau$. These are verified for the one-dimensional and the mean field cases. Thus, Nagatini's conjecture [11] that $(2-\tau_c)z_c=\frac{1}{4}$ is not true, and fails even for the mean field case.

For $\langle I\rangle>0$, the large negative clusters are statistically rare and physically one would expect that the steady state distribution $p(s)$ for large negative s would be cut off by a length $\xi(\langle I\rangle)$. This length should diverge as $\xi\sim\langle I\rangle^{-\nu}$ at the critical point $\langle I\rangle=0$, giving rise to a power law tail. It then seems reasonable to conjecture a dynamical scaling form for $p(s,t)$, for large negative s, large t, and small $\langle I\rangle$. A consistent scaling ansatz is

$$p(s,t)\sim|s|^{-\tau_c}f_-(|s|/t^{z_c},|s|/\xi). \tag{10}$$

When $\langle I\rangle\to0$, (10) is equivalent to (1), as expected, since $p(s,t)$ is then an even function. For finite positive $\langle I\rangle$, and for large time, the scaling function tends to a steady state function $|s|^{-\tau_c}f_-(0,|s|/\xi)$ with finite moments. This leads to $\langle s_-\rangle\sim\xi^{2-\tau_c}$. The form (10) also suggests that the crossover time is $\hat{t}=\xi^{1/z_c}\sim\langle I\rangle^{-\nu/z_c}$. Assuming that the crossover times scale identically with $\langle I\rangle$ for $s>0$ and $s<0$, we get $\nu=z_c\phi=1$ due to the scaling relations derived earlier, and it does not depend on the dimension. The exponent ν derived from the two closest points to the critical point (among three) from Fig. 1 of [11] is compatible with our result $\nu=1$.

To summarize, we have solved exactly the dynamics of the generalized Takayasu model in one dimension and in the mean field limit, and determined the dynamic cluster-size and crossover exponents. In fact, our general solution in these cases yields the full dynamic scaling functions as well. In addition, assuming dynamic scaling in higher dimensions, we have found the scaling relations connecting all the exponents to a single steady state exponent, both for symmetric ($\langle I\rangle=0$) and asymmetric ($\langle I\rangle>0$) cases. Finally, consistent with all these results, we conjecture, for $\langle I\rangle>0$, a new dynamic scaling form for $p(s,t)$ for $s<0$ near the critical point. This also introduces a steady state scaling function for $s<0$, cut off by a length which diverges at $\langle I\rangle=0$. In this respect, $\langle I\rangle=0$ is truly a critical point in the conventional sense. It would be interesting to check this dynamic scaling for $s<0$. Preliminary numerical results seem to confirm this conjecture [14]. Finally, another important issue would be to determine the upper critical dimension above which the mean field exponents are exact. Numerical results in $d=2$, 3, and 4 at present are not decisive on this question [6].

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