Trapping of classical and quantum walkers

Piégeage de marcheurs classiques et quantiques

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Setup

Comparing the dynamics of a classical and of a quantum walker

For definiteness: Continuous-time dynamics on the chain

1. *Free propagation*

2. *One single trap*

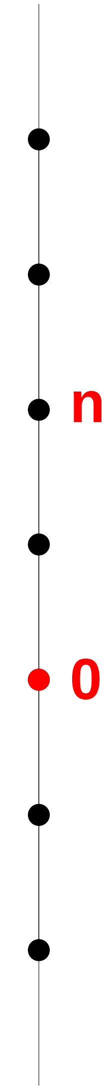
3. *Random distribution of traps*

P.L. Krapivsky, JML & K. Mallick, J. Stat. Phys. **154** (2014) 1430–1460

1. Free propagation

Classical walker: freely diffusing particle

Continuous-time random walk on infinite chain, starting from 0



- Master equation for classical probabilities

$$\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = p_{n+1}(t) + p_{n-1}(t) - 2p_n(t)$$

$$p_n(t) = \mathrm{e}^{-2t} I_n(2t) \approx \frac{\mathrm{e}^{-n^2/(4t)}}{\sqrt{4\pi t}}$$

- Asymptotic Gaussian profile and diffusive spreading

$$\langle n^2 \rangle = 2t, \qquad \langle n^4 \rangle = 12t^2 + 2t$$

- Strong recurrence: local time at 0

$$T_0(t) = \int_0^t p_0(t')\,\mathrm{d}t' = t\mathrm{e}^{-2t}(I_0(2t) + I_1(2t)) \approx \sqrt{\frac{t}{\pi}}$$

Quantum walker: freely propagating particle

Continuous-time quantum walk on infinite chain, starting from 0

- Tight-binding equation for wavefunction amplitudes

$$\mathrm{i}\,\frac{\mathrm{d}\psi_n(t)}{\mathrm{d}t} = \psi_{n+1}(t) + \psi_{n-1}(t)$$

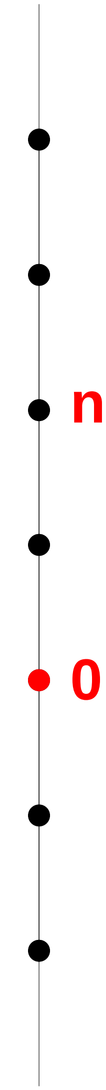
$$\psi_n(t) = \mathrm{i}^{-n} J_n(2t)$$

- Asymptotic ballistic spreading

$$\langle n^2 \rangle = 2t^2, \qquad \langle n^4 \rangle = 6t^4 + 2t^2$$

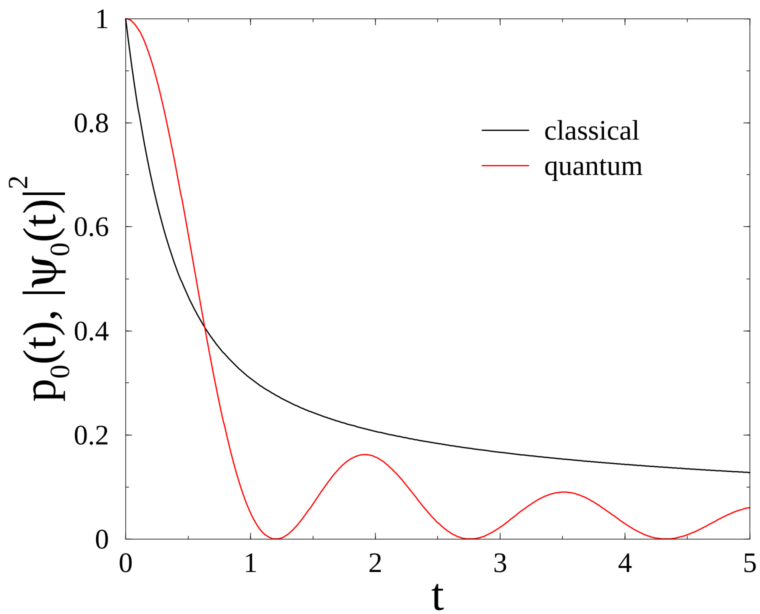
- Marginal recurrence: local time at 0

$$T_0(t) = \int_0^t \left|\psi_0(t')\right|^2 \mathrm{d}t' \approx \frac{\ln t}{2\pi}$$

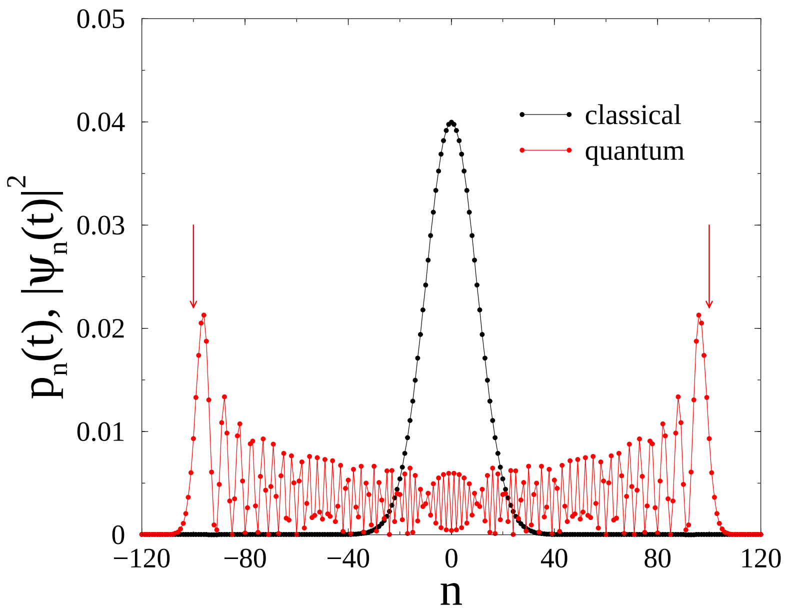


Comparing classical and quantum walker

in time



in space



Quantum wavepacket exhibits ballistic fronts near $n = \pm 2t$

$$|n| = 2t + t^{1/3} z, \qquad |\psi_n(t)|^2 \approx t^{-2/3} (\mathrm{Ai}(z))^2$$

An amazing consequence

S. de Toro Arias & JML (1998)

Dynamical participation ratios

$$S_q(t) = \sum_n |\psi_n(t)|^{2q}$$

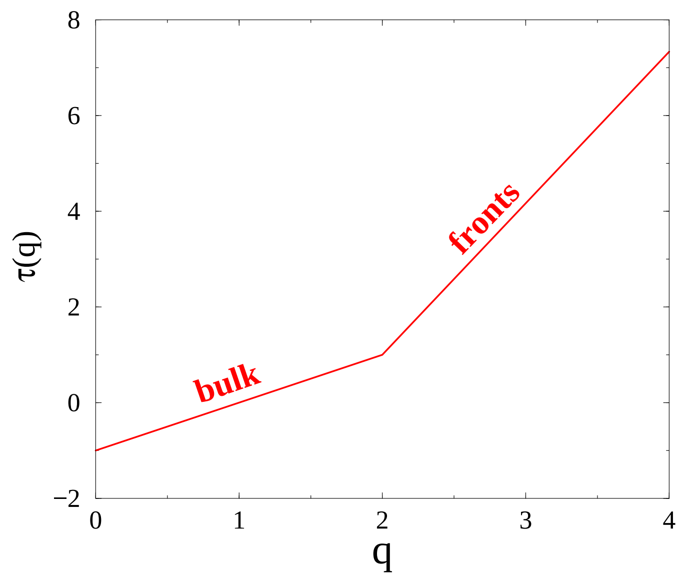
- Bifractal law of temporal decay

$$S_q(t) \sim t^{-\tau(q)}, \qquad \tau(q) = \begin{cases} q-1 & \text{for } q<2 \quad \text{(normal, bulk-driven)} \\ \frac{1}{3}(2q-1) & \text{for } q>2 \quad \text{(anomalous, front-driven)} \end{cases}$$

- In the presence of weak random potential

  same bifractal spectrum

  describes crossover to localized regime



2. One single trap

Single trap on the chain: classical case

Strength of classical trap is *absorption rate* γ *per unit time*

- Non-conservative master equation

$$\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = p_{n+1}(t) + p_{n-1}(t) - 2p_n(t) - \gamma\delta_{n0}\, p_n(t)$$

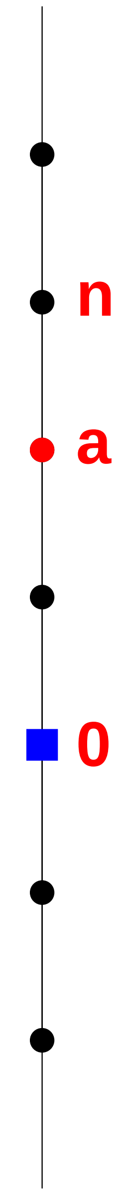
- Survival probability $\quad P(t) = \sum_n p_n(t) = 1 - \gamma \int_0^t p_0(t')\,\mathrm{d}t'$

- Exact expression in Laplace space

$$s\widehat{P}(s) = 1 - \frac{\gamma}{\gamma + \sqrt{s(s+4)}} \left(\frac{s + 2 - \sqrt{s(s+4)}}{2} \right)^a$$

- *Universal power-law decay* $\quad P(t) \approx \frac{b}{\sqrt{\pi t}}$

- *Trapping strength renormalizes distance* $\quad b = a + \frac{2}{\gamma}$



Single trap on the chain: quantum case

Strength of quantum trap is *amplitude* γ *of local optical potential*

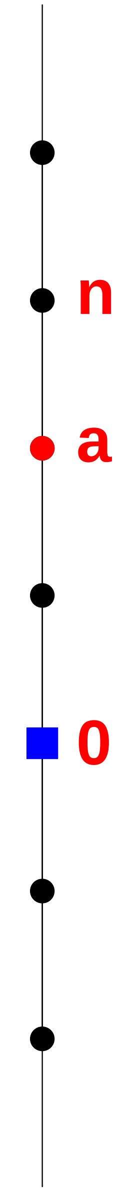
- Non-unitary tight-binding equation

$$\mathrm{i}\,\frac{\mathrm{d}\psi_n(t)}{\mathrm{d}t} = \psi_{n+1}(t) + \psi_{n-1}(t) - \mathrm{i}\gamma\delta_{n0}\,\psi_n(t)$$

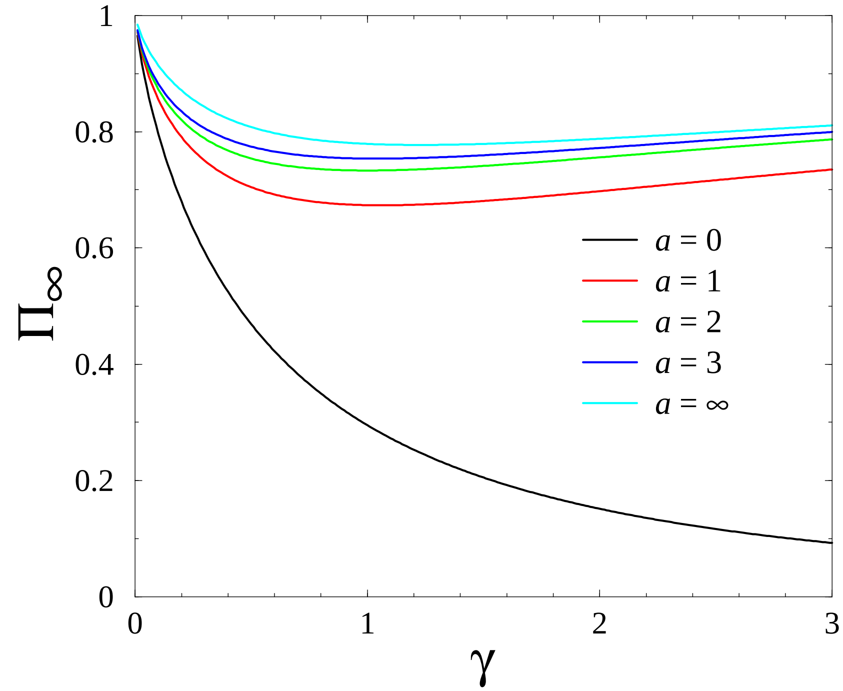
- Survival probability $\Pi(t) = \sum_n |\psi_n(t)|^2 = 1 - 2\gamma \int_0^t \left|\psi_0(t')\right|^2 \mathrm{d}t'$

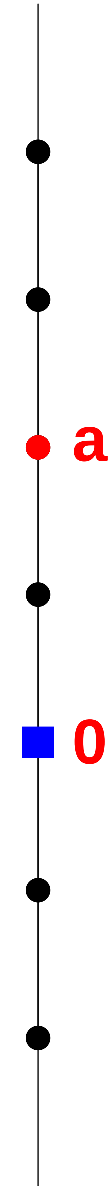
- *Non-trivial asymptotic value*

$$\begin{aligned}\Pi_\infty &= 1 - 2\gamma \int_0^\infty |\psi_0(t)|^2\,\mathrm{d}t \\ &= 1 - \frac{2\gamma}{\pi}\left(\int_0^2 \frac{\mathrm{d}x}{(\gamma + \sqrt{4-x^2})^2} + \int_2^\infty \frac{\mathrm{d}x}{\gamma^2 + x^2 - 4}\left(\frac{x - \sqrt{x^2-4}}{2}\right)^{2a}\right)\end{aligned}$$



Features of asymptotic survival probability of quantum walker





- $a = 0$ *Monotonic behavior*

$$\Pi_\infty \approx \frac{1}{2\gamma^2}$$

- $a > 0$ *Non-monotonicity effects*

Non-trivial minimum (maximal trapping efficiency)

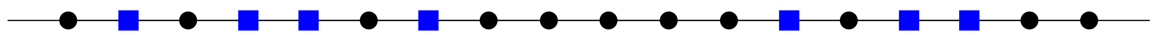
Paradoxical return: $$\Pi_\infty \approx 1 - \frac{16a^2}{(4a^2 - 1)\pi\gamma}$$

'Explanation': $$\psi_0(t) \approx \frac{a\,\mathrm{i}^{-a}}{\gamma}\,\frac{J_a(2t)}{t}$$

3. Random distribution of traps

Disordered system: classical case

- Binary (dilution) disorder



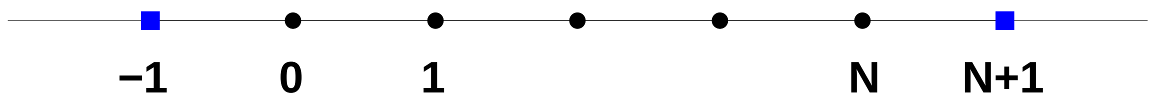
$$\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = p_{n+1}(t) + p_{n-1}(t) - 2p_n(t) - \gamma \varepsilon_n p_n(t)$$

$$\varepsilon_n = \begin{cases} 1 & \text{(trap)} \qquad \text{with prob.} \quad c \\ 0 & \text{(no trap)} \quad \text{with prob.} \quad 1-c \end{cases}$$

- Decay of survival probability (averaged over uniform initial point)

Lifshitz theory   Lifshitz (1964)

- Stationary problem: $p_n(t) \sim \mathrm{e}^{-\lambda t}$
- Cluster of $N+1$ sites without traps



$$p_n = A\,\mathrm{e}^{inq} + B\,\mathrm{e}^{-inq}, \qquad \lambda = 2(1-\cos q)$$

- Boundary conditions: $p_{-1} = Y_L\, p_0, \qquad p_{N+1} = Y_R\, p_N$
- Lowest mode of large cluster

$$q_1 = \frac{\pi}{N}\left(1 + \frac{\alpha}{N} + \cdots\right), \qquad \alpha = \frac{1}{Y_L - 1} + \frac{1}{Y_R - 1}$$

- Decay rate scales *diffusively* and is *deterministic* (i.e., independent of b.c.)

$$\lambda_1 \approx \frac{\pi^2}{N^2}$$

To sum up

- Large cluster has both

  Small probability $(1-c)^N$

  Long survival time: Decay rate $\dfrac{\pi^2}{N^2}$ (irrespective of b.c.)

- Average survival probability

$$P(t) \sim \int_0^\infty \exp\left(-\frac{\pi^2}{N^2}t - |\ln(1-c)|N\right) \mathrm{d}N$$

- Saddle point (*Optimal* cluster size) $N \approx \left(\dfrac{2\pi^2}{|\ln(1-c)|}t\right)^{1/3}$

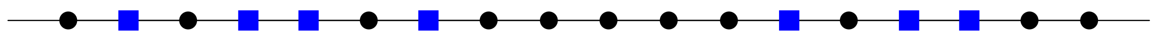
- Stretched exponential decay as the result of a compromise

$$P(t) \sim \exp\left(-\frac{3}{2}\left(2\pi^2 |\ln(1-c)|^2 t\right)^{1/3}\right)$$

Balagurov & Vaks (1974), Donsker & Varadhan (1975),
Grassberger & Procaccia (1982)

Disordered system: quantum case

Binary (dilution) disorder



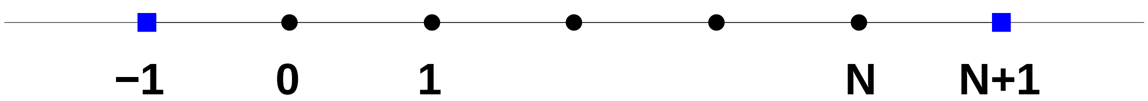
$$\mathrm{i}\,\frac{\mathrm{d}\psi_n(t)}{\mathrm{d}t} = \psi_{n+1}(t) + \psi_{n-1}(t) - \mathrm{i}\gamma\varepsilon_n\,\psi_n(t)$$

$$\varepsilon_n = \begin{cases} 1 & \text{(trap)} \quad \text{with prob.} \quad c \\ 0 & \text{(no trap)} \quad \text{with prob.} \quad 1-c \end{cases}$$

Parris (1989), Edwards & Parris (1989)

Approach à la Lifshitz

- Stationary problem: $\psi_n(t) \sim \mathrm{e}^{-\mathrm{i}Et}, \qquad |\psi_n(t)|^2 \sim \mathrm{e}^{2(\,\mathrm{Im}\,E)t}$
- Cluster of $N+1$ sites without traps



$$\psi_n = A\,\mathrm{e}^{inq} + B\,\mathrm{e}^{-inq}, \qquad E = 2\cos q$$

- Boundary conditions: $\psi_{-1} = Y_L \psi_0, \qquad \psi_{N+1} = Y_R \psi_N$
- Lowest mode of large cluster

$$q_1 = \frac{\pi}{N}\left(1 + \frac{\alpha}{N} + \cdots\right), \qquad \alpha = \frac{1}{Y_L - 1} + \frac{1}{Y_R - 1}$$

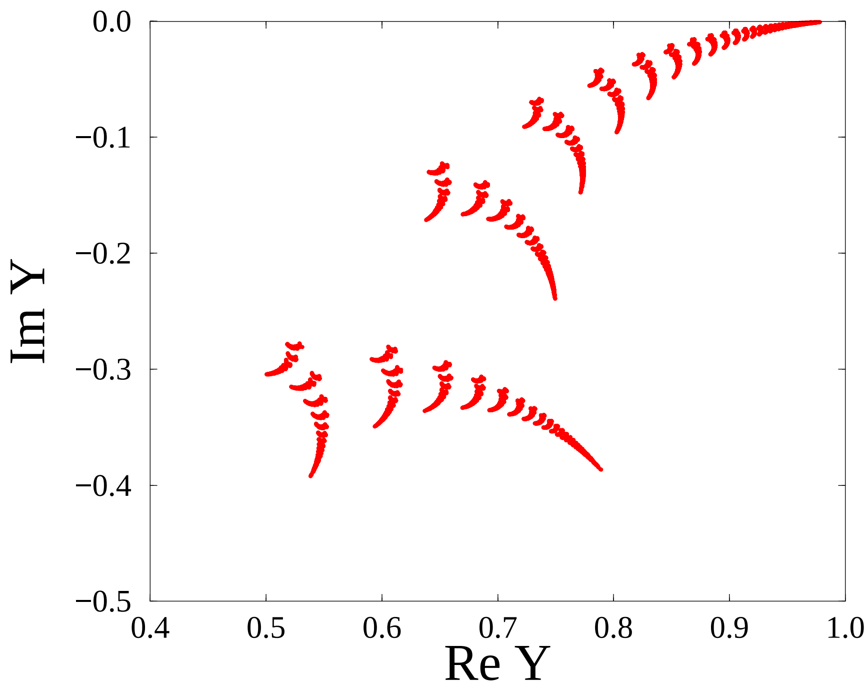
- Decay rate now scales as $1/N^3$ and *fluctuates* (i.e., depends on b.c.)

$$\lambda = -2\;\mathrm{Im}\;E_1 \approx 2\;\mathrm{Im}\;q_1^2 \approx \frac{4\pi^2}{N^3}\;\mathrm{Im}\;\alpha$$

How is $\text{Im}\,\alpha$ distributed? What is its minimum?

Clue: Riccati variables $Y_n = \frac{\psi_n}{\psi_{n+1}}$ at band edge ($q = 0$, i.e., $E = 2$)

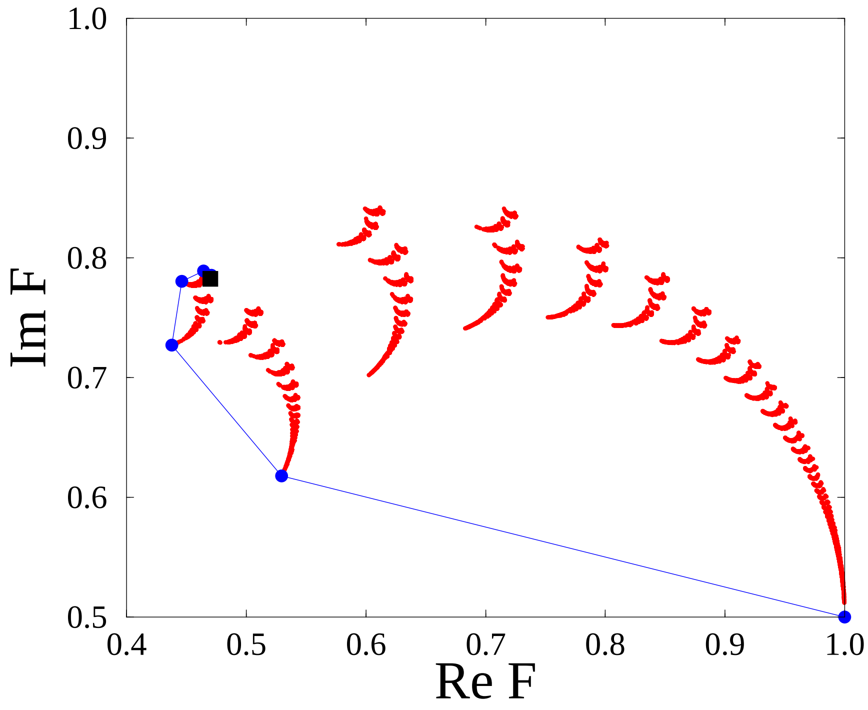
- Obey recursion $$Y_n = \frac{1}{2 + i\gamma\varepsilon_n - Y_{n-1}}$$
- Have limit distribution whose support is complex fractal
- Boundary parameters Y_L and Y_R are independent r.v. with this law



$(\gamma = \frac{1}{2})$

New variables: $F_n = \dfrac{i\gamma}{1 - Y_n}$ obey recursion $F_n = i\gamma + \dfrac{F_{n-1}}{1 + \varepsilon_n F_{n-1}}$

- $\min(\,\mathrm{Im}\,\alpha) = \dfrac{2f(\gamma)}{\gamma}$ $\min(\lambda) \approx \dfrac{8\pi^2}{N^3}\dfrac{f(\gamma)}{\gamma}$
- $f(\gamma) = \min(\,\mathrm{Re}\,F\,)$
- Only depends on γ. Reached for some point of blue sequence

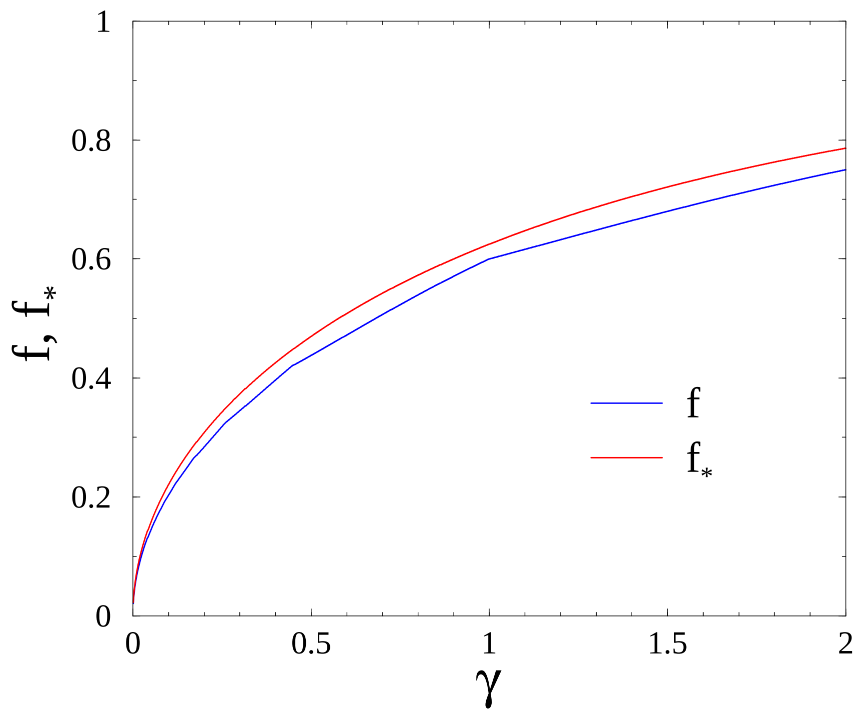


Blue sequence: Extremal points F_k

$$F_k = \mathrm{i}\gamma + \frac{F_{k-1}}{1+F_{k-1}} \qquad (F_0 = \infty)$$

$$\gamma > 1: \qquad k(\gamma) = 2, \qquad f(\gamma) = \ \mathrm{Re}\, F_2 = \frac{\gamma^2+2}{\gamma^2+4}$$

$$\gamma \ll 1: \qquad k(\gamma) \approx \frac{\pi}{\sqrt{2\gamma}}, \qquad f(\gamma) \approx \underbrace{\tanh\frac{\pi}{2}}_{0.917152} \sqrt{\frac{\gamma}{2}}$$



To sum up

- Large cluster has

  Probability $(1-c)^N$

  Optimal decay rate $\frac{8\pi^2}{N^3}\frac{f(\gamma)}{\gamma}$

- Average survival probability of quantum walker

$$\Pi(t) \sim \int_0^\infty \exp\left(-\frac{8\pi^2}{N^3}\frac{f(\gamma)}{\gamma}t - |\ln(1-c)|N\right) \mathrm{d}N$$

- Saddle point (*Optimal* cluster size) $N \approx \left(24\pi^2 \frac{f(\gamma)}{\gamma}\frac{t}{\ln(1-c)}\right)^{1/4}$

- Stretched exponential decay

$$\Pi(t) \sim \exp\left(-\frac{4}{3}\left(24\pi^2 \frac{f(\gamma)}{\gamma}|\ln(1-c)|^3 t\right)^{1/4}\right)$$

Only exponent was known so far Parris (1989), Edwards & Parris (1989)

Outline

- Various facets of qualitatively different behavior of classical and quantum walkers

- Survival in the presence of random distribution of traps

Classical: $P(t) \sim \exp\left(-A_d\, t^{d/(d+2)}\right)$

A_d predicted by Lifshitz theory

Optimize size of Lifshitz sphere (b.c. do not matter)

Quantum: $\Pi(t) \sim \exp\left(-B_d\, t^{d/(d+3)}\right)$

B_d more subtle... even for $d = 1$

Optimize size and *boundary conditions*

Riccati variables at work

To close ...

Disorder in one dimension is an inexhaustible source of pleasure

(freely adapted from F. Dyson)

Happy Birthday Alain !...