RENORMALIZATION

FOR

DISORDERED SYSTEMS

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Prologue : Merci à Alain Comtet !

Professeur au DEA de Physique Théorique

- J'ai fait la connaissance d'Alain Comtet en Septembre 1991
- Stage de D.E.A. en Janvier 1992

Thèse (1992 - 1995) : "Etude de quelques fonctionnelles du mouvement Brownien et de certaines propriétés de la diffusion unidimensionnelle en milieu aléatoire"

1. Propriétés d'enroulement du mouvement Brownien plan,
   par le formalisme de l'intégrale de chemin (Feynman-Kac)
2. Diffusion anormale en milieu aléatoire
   par l'étude de fonctionnelles exponentielles du mouvement Brownien
3. Propriétés de localisation pour un Hamiltonien quantique désordonné 1D

→ Ces travaux m'ont permis d'apprendre les 'bases' sur :

- les probabilités, le mouvement Brownien et les processus stochastiques
- les systèmes désordonnés, à la fois classiques et quantiques

Disorder → spatial heterogeneities

First example where disorder changes the physics : Anderson localization (1958)

- Perfect cristal with translation invariance → delocalized wave functions
- Disorder breaking translation invariance

→ localized wavefunction in $d \leq 2$ and localization transition in $d > 2$

Some famous probabilistic arguments based on disorder fluctuations

1. concerning "Rare events" :
   - Lifshitz argument (1964)
     → essential singularity for the density of states near the spectrum edge
   - Griffiths phases (1969)
     → influence of rare ordered regions in the disordered phase.
2. concerning "Typical events" :
   - Harris criterium (1974)
     → relevance of disorder at a pure critical point.
   - Imry-Ma argument (1975)
     → lower critical dimension for random field systems.

→ Real Space RG better to describe these disorder spatial heterogeneities

Renormalization in disordered systems

Renormalization → emergence of universal large scale properties

Renormalization in pure systems (without disorder)

- small number of relevant couplings
- focus on critical points (the ordered and disordered phases are 'clear')
- translation invariance : many RG procedures defined in Fourier space

Renormalization in the presence of quenched disorder

- one needs to renormalize probability distributions (space of ∞ dimensions)
  → much more difficult to determine the fixed points
- before the phase transition towards the high-T disordered phase,
  one needs to understand the properties of the low-T frozen-phase, governed by the non-trivial zero-temperature fixed point
  Ex : spin-glass → what are the properties of the 'spin-glass phase' ?
- translation invariance is broken : real space RG procedures are more suited

From the point of view of probability theory : Renormalization on random variables → Fixed point for an appropriate rescaled variable

First simple example : sum of independent random variables !

→ Central Limit Theorem : exponents, stable laws, attraction bassins...

Droplet scaling theory of the spin-glass phase
Mc Millan (1984) ; Bray and Moore (1986) ; Fisher and Huse (1986) ...

RG flow for the distribution of the renormalized couplings J_L at scale L :

$$P_L(J_L) \underset{L\to\infty}{\simeq} \frac{1}{L^\theta} \mathcal{P}^* \left(\frac{J_L}{L^\theta} \right)$$

Physical meaning : free-energy cost of an interface (Ferromagnets $\theta = d - 1$)

- $d = 2$: Exponent $\theta < 0$ → disorder becomes weaker and weaker
  → no spin-glass phase at finite temperature
- $d = 3$: Exponent $\theta > 0$ → disorder becomes stronger and stronger
  → spin-glass phase in a finite region $[0, T_c[$ of temperature

Long-ranged Spin-Glass with power-law interaction

Real spin-glasses with RKKY interactions $J^{RKKY}(r) \simeq \pm \frac{1}{r^3}$

Long-ranged Spin-Glass with power-law interaction of exponent σ

$$E(S_1, ..., S_L) = - \sum_{1 \leq i < j \leq L} J_{ij} S_i S_j$$

Random Couplings $J_{ij} = \frac{\epsilon_{ij}}{|j-i|^\sigma}$ where the ϵ_{ij} are independent identical $O(1)$ random variables of zero mean.

- Gaussian distribution

$$L_2(\epsilon) = \frac{1}{\sqrt{4\pi}} e^{-\frac{\epsilon^2}{4}}$$

- Lévy symmetric stable law $L_\mu(\epsilon)$ of index $1 < \mu \leq 2$

$$L_\mu(\epsilon) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{-ik\epsilon - |k|^\mu}$$

Extensivity of the ground state energy for $\sigma > \frac{1}{\mu}$
(in particular $\sigma > \frac{1}{2}$ for the Gaussian case $\mu = 2$)

Explicit expression for the droplet exponent θ^{LR}

For Short-Ranged Spin-Glasses : except in one dimension $\theta^{SR}(d=1)=-1$, the droplet exponent is only known numerically in $d>1$.

Scaling argument for the Gaussian LR case (Bray, Moore, Young 1986, Fisher, Huse 1988)

$$\theta^{LR}_{Gauss}(\sigma) = 1-\sigma \quad \text{for} \quad \frac{1}{2}<\sigma<2$$

$$\theta^{LR}_{Gauss}(\sigma) = \theta^{SR}(d=1) = -1 \quad \text{for} \quad 2 \leq \sigma$$

Scaling argument for the Lévy LR case

$$\theta^{LR}_{\mu}(\sigma) = \frac{2}{\mu}-\sigma \quad \text{for} \quad \frac{1}{\mu}<\sigma<\frac{2}{\mu}+1$$

$$\theta^{LR}_{\mu}(\sigma) = \theta^{SR}(d=1) = -1 \quad \text{for} \quad \sigma > \frac{2}{\mu}+1$$

Renormalization at zero temperature
C. Monthus, J. Stat. Mech. P06015 (2014)

Simplest decimation using blocks of size $b = 2$

Minimization of the internal energy of each block $E^{int}_{2n-1,2n} = -J_{2n-1,2n} S_{2n-1} S_{2n}$

$$S_{2n-1} = S_{2n} \mathrm{sign}(J_{2n-1,2n})$$

Energy $E = -\sum_n |J_{2n-1,2n}| - \sum_{-\infty \leq n < m \leq +\infty} J^{(1)}_{2n,2m} S_{2n} S_{2n}$
with the renormalized couplings between the remaining even spins

$$\begin{aligned} J^{(1)}_{2n,2m} &= J_{2n,2m} + \mathrm{sgn}(J_{2n-1,2n}) \mathrm{sgn}(J_{2m-1,2m}) J_{2n-1,2m-1} \\ &+ \mathrm{sgn}(J_{2n-1,2n}) J_{2n-1,2m} + \mathrm{sgn}(J_{2m-1,2m}) J_{2n,2m-1} \end{aligned}$$

$\rightarrow$ same Lévy stable law L_μ, with the renormalized characteristic scale

$$\Delta^{(1)}(2r) = [2\Delta^\mu(2r) + \Delta^\mu(2r+1) + \Delta^\mu(2r-1)]^{\frac{1}{\mu}}$$

Iteration $\rightarrow$ correct droplet exponent $\theta^{LR}_\mu(\sigma) = \frac{2}{\mu} - \sigma$ only if positive.

Renormalization at zero temperature

Improved procedure : Strong Disorder Decimation

The odd spin S_{2n-1} is associated to its left S_{2n-2} or to its right neighbor S_{2n} depending on the biggest absolute coupling between $J_{2n-2,2n-1}$ and $J_{2n-1,2n}$.

$$\begin{aligned} S_{2n-1} &= \theta(|J_{2n-1,2n}| - |J_{2n-2,2n-1}|)\mathrm{sgn}(J_{2n-1,2n})S_{2n} \\ &+\theta(|J_{2n-2,2n-1}| - |J_{2n-1,2n}|)\mathrm{sgn}(J_{2n-2,2n-1})S_{2n-2} \end{aligned}$$

• Exactness for the nearest-neighbor spin-glass chain $(\sigma = +\infty)$:
$\theta^{SR}_{\mu}(\sigma = +\infty) = -1$
• Exactness for the Migdal-Kadanoff approximation (diamond fractal hierarchical lattice)

Conclusion for Short-Ranged SG :
the block decimation gives a too large upper bound $\theta^{SR}_{block}(d) = \frac{d-1}{2}$
$\longrightarrow$ necessary to introduce an appropriate generalization of the Strong Disorder decimation in $d > 1$

Statistics over samples of the ground state energy of the LR Spin-Glass obtained by the RG procedure

Averaged value

$$\overline{E^{GS}(L)} \simeq L e_0 + L^{\theta_{shift}} e_1 + \dots$$

The first term $L^d e_0$ is the extensive contribution
The second term $L^{\theta_{shift}} e_1$ representing the leading correction to extensivity is governed by the droplet exponent $\theta_{shift} = \theta$ (as in Short-Ranged SG) .

Fluctuations around the averaged value

$$E^{GS}(L) - \overline{E^{GS}(L)} \simeq L^g u + \dots$$

where u is an $O(1)$ random variable of zero mean $\overline{u} = 0$ distributed with some distribution $G(u)$.

- Gaussian couplings : $g = \frac{1}{2}$ and $G(u)$ is Gaussian
(as in Short-Ranged Spin-Glasses : $g = \frac{d}{2}$ in any finite d: Wehr-Aizenman 1990)
- Lévy couplings : $g = \frac{1}{\mu}$ and power-law tail in $G(u \to -\infty) \propto 1/(-u)^{1+\mu}$

Barrier Exponent ψ for the dynamics

Low-temperature dynamics starting from a random initial condition

• Interpretation in terms of a growing 'coherence length' $l(t)$:
the smaller lengths $l < l(t)$ are quasi-equilibrated
the bigger lengths $l > l(t)$ are completely out of equilibrium
Equilibrium is reached only when $l(t_{eq}) = L =$ the system size.

• Dynamics is extremely slow because equilibration on larger length scales requires to overcome larger and larger barriers.
The barriers grow as a power of the length l with some barrier exponent $\psi > 0$

$$B(l) \equiv \ln t_{typ}(l) \sim l^{\psi}$$

ACTIVATED SCALING with a universal exponent ψ : $l(t) \sim (\ln t)^{\frac{1}{\psi}}$
Example of the diffusion in a Brownian random potential (Sinai model) :
$\psi = 1/2$ leading to logarithmically-slow motion $l(t) \sim (\ln t)^2$
completely different from the pure diffusion $l(t) \sim t^{\frac{1}{2}}$

SPIN-GLASSES : the activated dynamics is also completely different from the dynamics in pure ferromagnets $l(t) \propto t^{1/z}$ with the dynamical exponent $z = 2$ for non-conserved dynamics (domain walls diffuse and annihilate)

Relaxation towards thermal equilibrium

Classical system where each configuration $\mathcal{C}$ has some energy $U(C)$

Stochastic dynamics described by a Master Equation

Evolution of the probability $P_t(\mathcal{C})$ to be in configuration $\mathcal{C}$ at time t :

$$\frac{dP_t(\mathcal{C})}{dt} = \sum_{\mathcal{C}'} P_t(\mathcal{C}') W(\mathcal{C}' \to \mathcal{C}) - P_t(\mathcal{C}) W_{out}(\mathcal{C})$$

- $W(\mathcal{C}' \to \mathcal{C})$ represents the transition rate per unit time from $\mathcal{C}'$ to $\mathcal{C}$
- $W_{out}(\mathcal{C}) \equiv \sum_{\mathcal{C}'} W(\mathcal{C} \to \mathcal{C}')$ represents the total exit rate out of $\mathcal{C}$.

The Detailed Balance property $e^{-\beta U(\mathcal{C})} W(\mathcal{C} \to \mathcal{C}') = e^{-\beta U(\mathcal{C}')} W(\mathcal{C}' \to \mathcal{C})$

ensures the convergence towards Boltzmann equilibrium $P_{eq}(\mathcal{C}) = \frac{e^{-\beta U(\mathcal{C})}}{Z}$
Example : Metropolis single-spin-flip dynamics $S_k \to -S_k$

$$W(\mathcal{C} \to \mathcal{C}_k) = \frac{1}{\tau_0} \min\left[1, e^{-\beta[U(\mathcal{C}_k) - U(\mathcal{C})]}\right]$$

Renormalization for the dynamics of the LR Spin-Glass C. Monthus, J. Stat. Mech. P08009 (2014)

Renormalization of the transition rates : Full hierarchy of relaxation times

Closed RG for the generalized Metropolis dynamics, where each spin S_k has its own characteristic time τ_k to attempt a spin-flip

$$W\left(\mathcal{C} \to \mathcal{C}_k\right) = \frac{1}{\tau_k} \min\left[1, e^{-\beta[U(\mathcal{C}_k)-U(\mathcal{C})]}\right]$$

- First RG step $\tau_{S^{R1}_{2i}} = \tau_0 e^{2\beta|J_0(2i-1,2i)|}$
- Second RG step $\tau_{S^{R2}_{4i}} = e^{2\beta|J^{(R1)}_1(4i-2,4i))|} \frac{\tau_{S^{R1}_{4i-2}}+\tau_{S^{R1}_{4i}}}{2}$
- Last RG step $\tau_{S^{RN}_{2^N}} = e^{2\beta|J^{(R(N-1))}_{N-1}(2^{N-1},2^N))|} \frac{\tau_{S^{R(N-1)}_{2^{N-1}}}+\tau_{S^{R(N-1)}_{2^N}}}{2}$

Result for the dynamical exponent ψ

- The last renormalized coupling $|J^{(R(N-1))}_{N-1}|$ yields the usual bound $\psi \geq \theta$
- Gaussian couplings $\mu = 2$: $\psi_2(\sigma) = \theta_2(\sigma) = 1 - \sigma$
- Lévy couplings $1 < \mu < 2$: $\psi_\mu(\sigma) = \frac{1}{\mu} > \theta_\mu(\sigma) = \frac{2}{\mu} - \sigma$

Conclusion : the Long-Ranged Spin-Glass Chain as a simple example of the droplet scaling theory

Statics at zero temperature $\longrightarrow$ droplet exponent θ

- Explicit renormalization of the couplings at zero-temperature
- Explicit expression for the droplet exponent θ
- Consequences for the statistics over samples of the ground state energy (θ governs the leading correction to extensivity of the averaged value).

Dynamics near zero-temperature $\longrightarrow$ barrier exponent ψ

- Explicit renormalization of the transition rates near zero-temperature.
- The convergence towards local equilibrium on larger and larger scales is governed by a strong hierarchy of activated dynamical processes, with valleys within valleys

Perspectives

- Extend the RG to finite-T to study the Spin-Glass/Paramagnet transition.
- Define an appropriate RG procedure for the Short-Ranged Spin-Glass in $d > 1$