Fluctuations of certain random matrix products

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The group $\mathrm{SL}(2,\mathbb{R})$ and the Iwasawa representation

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \in \mathrm{SL}(2,\mathbb{R})$$

with

$$a,\ b,\ c,\ d\ \in \mathbb{R} \qquad \& \qquad ad - bc = 1$$

3 independent real parameters

Iwasawa :

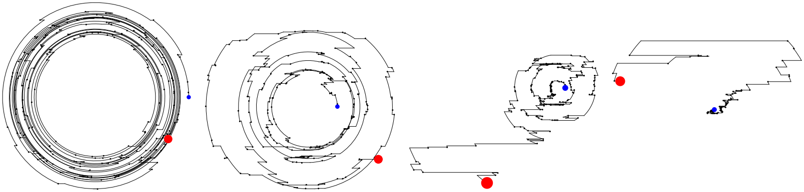
$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{w} & 0 \\ 0 & e^{-w} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

Example 1

$$\Pi_N = M_N \cdots M_1 \quad \text{with} \quad M_n = \begin{pmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{pmatrix} \begin{pmatrix} 1 & u_n \\ 0 & 1 \end{pmatrix}$$

Motion of $\begin{pmatrix} x_N \\ y_N \end{pmatrix} = \Pi_N \begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

Scale $\theta_n \propto k$ and $u_n \propto 1/k$

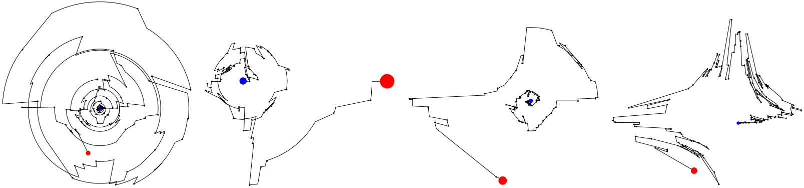


$k = 5$ $\quad$ $k = 2$ $\quad$ $k = 1$ $\quad$ $k = 0.5$

Example 2

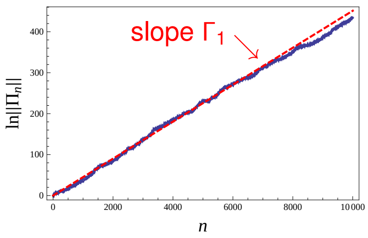
$$\Pi_N = M_N \cdots M_1 \quad \text{with} \quad M_n = \begin{pmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{pmatrix} \begin{pmatrix} e^{w_n} & 0 \\ 0 & e^{-w_n} \end{pmatrix}$$

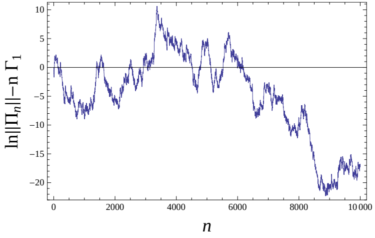
Scale $\theta_n \propto k$ and $w_n \propto k^0$



$k = 5$ $k = 2$ $k = 1$ $k = 0.2$

Fluctuations of $\ln ||\Pi_n||$





Generalised central limit theorem :

Bougerol & Lacroix, (the book, chapter V) 1985

$$\Pi_n = M_n \cdots M_2 M_1 \qquad (\text{non commuting})$$

$$\Gamma_1 = \lim_{n\to\infty} \frac{1}{n} \ln \left|\left| \Pi_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right|\right| \qquad \& \qquad \Gamma_2 = \lim_{n\to\infty} \frac{1}{n} \mathrm{Var}\left(\ln \left|\left| \Pi_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right|\right| \right)$$

Lyapunov exponent Γ_1 for matrices of $\mathrm{SL}(2,\mathbb{R})$

Q : given a measure $\mu(\mathrm{d}M)$ in $\mathrm{SL}(2,\mathbb{R})$,

$$\text{what is} \quad \Gamma_1 = \lim_{N\to\infty} \frac{\ln ||\Pi_N||}{N} \ ?$$

$\rightarrow$ few solvable cases

Density of θ	Density of u	Density of w	Reference
$\delta(\theta - 1/\rho)$	$\frac{q/\pi}{q^2+u^2}$	$\delta(w)$	Ishii '73
$\mathbf{1}_{\mathbb{R}_+}(\theta)\,\rho\,e^{-\rho\theta}$	$\mathbf{1}_{\mathbb{R}_+}(u)\,q\,e^{-qu}$	$\delta(w)$	N '83 ; CTT '10 (*)
$\mathbf{1}_{\mathbb{R}_+}(\theta)\,\rho\,e^{-\rho\theta}$	$\mathbf{1}_{\mathbb{R}_+}(u)\,q^2 u\,e^{-qu}$	$\delta(w)$	N '83 ; CTT '10
$\mathbf{1}_{\mathbb{R}_+}(\theta)\,\rho\,e^{-\rho\theta}$	$\delta(u)$	$\mathbf{1}_{\mathbb{R}_+}(w)\,q\,e^{-qw}$	CTT '10
$\mathbf{1}_{\mathbb{R}_+}(\theta)\,\rho\,e^{-\rho\theta}$	$\delta(u)$	$q\,e^{-2q\|w\|}$	CTT '12

(*) N $\equiv$ Nieuwenhuizen ; CTT $\equiv$ **Comtet**, Texier & Tourigny

Continuum limit : $M_n \to \mathbf{1}_2$ (i.e. $\theta_n,\ w_n,\ u_n \to 0$)

$$M_n \simeq \mathbf{1}_2 + \theta_n\,\Gamma_K + w_n\,\Gamma_A + u_n\,\Gamma_N$$

$$\Gamma_K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \Gamma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \Gamma_N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\mu(\mathrm{d}M) \longrightarrow P(\theta, w, u)$ Gaussian

General solution :

$$\Omega = \Gamma_1 - \mathrm{i}\pi\, j = \frac{\mathrm{d}}{\mathrm{d}(\cdot)} \ln(\text{special function})$$

A. Comtet, J.-M. Luck, C. Texier & Y. Tourigny, J. Stat. Phys. **150** (2013)

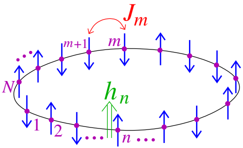
Q : Explicit formulae for Γ_2 ?

Outline

I. Physical motivations

Transfer matrices in statistical physics

Random Ising chain :



$$\mathcal{H}(\{\sigma_n\}) = -\sum_{n=1}^{N} \left(J_n\, \sigma_n \sigma_{n+1} + h_n\, \sigma_n \right)$$

$$\sigma_n = \pm 1$$

Transfer matrix approach :

$$\mathcal{Z}_N = \sum_{\{\sigma_n\}} e^{-\mathcal{H}(\{\sigma_n\})} = \mathrm{Tr}\left\{ T_N \cdots T_1 \right\}$$

with $T_n \stackrel{\text{def}}{=} \underbrace{\begin{pmatrix} e^{+h_n} & 0 \\ 0 & e^{-h_n} \end{pmatrix} \begin{pmatrix} e^{+J_n} & e^{-J_n} \\ e^{-J_n} & e^{+J_n} \end{pmatrix}}_{M_n = \frac{T_n}{\sqrt{2\sinh 2J_n}} \in \text{subgroup of } \mathrm{SL}(2,\mathbb{R})}$

Free energy :

$$\mathcal{F}_N = -\ln \mathcal{Z}_N = -\ln || T_N \cdots T_2 T_1 ||$$

→ mean : Γ_1

→ variance : Γ_2

Quantum localisation – Fluctuations

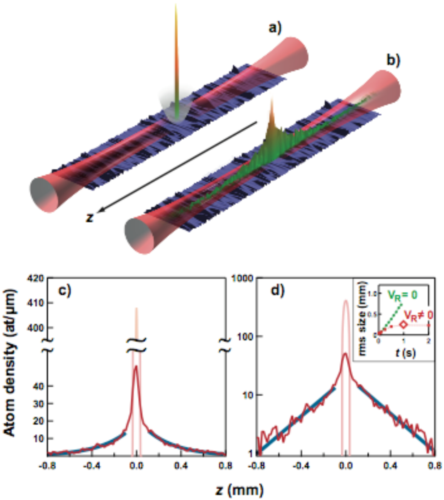
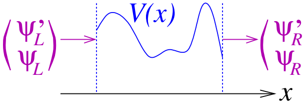


Figure 1. Observation of exponential localization. a) A small BEC (1.7 x 10^4 atoms) is formed in a hybrid trap, which is the combination of a horizontal optical waveguide ensuring a strong transverse confinement, and a loose magnetic longitudinal trap. A weak disordered optical potential, transversely invariant over the atomic cloud, is superimposed (disorder amplitude V_R small compared to the chemical potential μ_{in} of the atoms in the initial BEC). **b)** When the longitudinal trap is switched off, the BEC starts expanding and then localises, as observed by direct imaging of the fluorescence of the atoms irradiated by a resonant probe. On a and b, false colour images and sketched profiles are for illustration purpose, they are not exactly on scale. **c-d)** Density profile of the localised BEC, 1s after release, in linear or semi-logarithmic coordinates. The inset of Fig d (rms width ot the profile vs time t, with or without disordered potential) shows that the stationary regime is reached after 0.5 s. The diamond at t=1s corresponds to the data shown in Fig c-d. Blue solid lines in Fig c are exponential fits to the wings, corresponding to the straight lines of Fig d. The narrow profile at the centre represents the trapped condensate before release (t=0).

J. Billy et al. Nature **453**, 891 (2008)

$$-\psi''(x) + V(x)\,\psi(x) = k^2\,\psi(x)$$

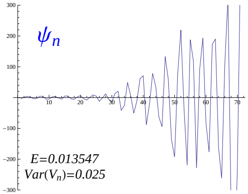
$$\begin{pmatrix}\psi'(x_R)\\ \psi(x_R)\end{pmatrix} = M \begin{pmatrix}\psi'(x_L)\\ \psi(x_L)\end{pmatrix}$$



Current conservation $\Rightarrow$ $M^\dagger \sigma_y M = \sigma_y \Rightarrow e^{i\theta}\, M \in \mathrm{SL}(2,\mathbb{R})$

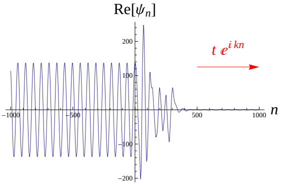
$$\begin{pmatrix}\psi'(x)\\ \psi(x)\end{pmatrix} = \Pi_N \begin{pmatrix}\psi'(0)\\ \psi(0)\end{pmatrix}$$

$$\boxed{\ln ||\Pi_N|| \leftrightarrow \ln |\psi(x)|}$$



Importance of fluctuations (1) : conductance

Conductance fluctuations

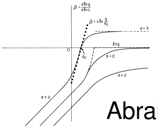


$\psi(x)$ the solution of $\begin{cases} \psi(0) = 0 \\ \psi'(0) = 1 \end{cases}$

For long sample : $\quad g = |t|^2 \sim |\psi(L)|^{-2}$

$$P(g;L) \simeq \frac{1}{g\sqrt{8\pi\gamma_2 L}} \exp -\frac{(\ln g + 2\gamma_1 L)^2}{8\gamma_2 L}$$

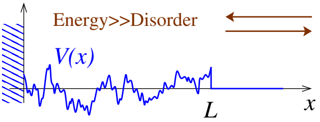
Single Parameter Scaling (SPS) hypothesis (weak disorder)



$$\gamma_2 = \lim_{x\to\infty} \frac{\mathrm{Var}(\ln|\psi(x)|)}{x} \simeq \gamma_1 = \lim_{x\to\infty} \frac{\ln|\psi(x)|}{x}$$

Abrahams, Anderson, Licciardello & Ramakrishnan, PRL **42** (1979)
Anderson, Thouless, Abraham & Fisher, PRB **22** (1980)
Shapiro, PRB **34** (1986)
etc.

Importance of fluctuations (2) : LDoS



Stationary scattering state :

$$\psi(x;k^2) \underset{x>L}{=} \frac{1}{\sqrt{2\pi v_k}}\left(e^{-\mathrm{i}k(x-L)} + e^{+\mathrm{i}k(x-L)+\mathrm{i}\,\eta(k^2)}\right)$$

$$v_k = \mathrm{d}E/\mathrm{d}k = 2k \text{ : group velocity}$$

Distribution of the LDoS $\rho(x;E) = \langle\, x\,|\delta(E-H)|\, x\,\rangle = |\psi(x;E)|^2$

$$P(\tilde{\rho};x) \underset{\text{disorder}}{\overset{\text{weak}}{\simeq}} \frac{1}{\tilde{\rho}\,\sqrt{8\pi\gamma_1(L-x)}}\ \exp -\frac{\left[\ln(\tilde{\rho}/\rho_0)+2\gamma_1(L-x)\right]^2}{8\gamma_1(L-x)}$$

where $\rho_0 = 1/(2\pi v_k)$

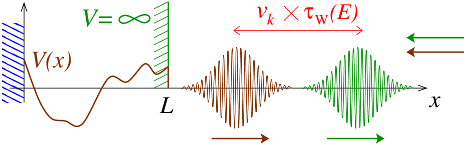
Berezinskii blocks method → Altshuler & Prigodin, Sov. Phys. JETP **68** (1989)

Importance of fluctuations (3) : Wigner time delay

$\eta(E)$: reflection phase shift

Wigner time delay

$$\tau_W(E) \stackrel{\text{def}}{=} \frac{\mathrm{d}\eta(E)}{\mathrm{d}E}$$



Exponential functional of the BM

$$\tau_W(E) \simeq 2\pi \int_0^L \mathrm{d}x \, \overbrace{\rho(x;E)}^{|\psi(x;E)|^2} \xrightarrow[\text{disorder}]{\text{weak}} \frac{1}{k}\int_0^L \mathrm{d}x \, e^{-2\gamma_1 x + 2\sqrt{\gamma_1}\,B(x)}$$

C. Texier & **A. Comtet**, PRL **82** (1999)

The model of interest today

Dirac equation with random mass

$$\left[\sigma_2\,\mathrm{i}\partial_x + \sigma_1\, m(x)\right]\Psi(x) = \varepsilon\,\Psi(x) \qquad \text{where } \Psi = \begin{pmatrix}\psi\\ \chi\end{pmatrix}$$

with

$$\langle m(x)m(x')\rangle_c = g\,\delta(x-x')$$

Motivations

relation to

- Supersymmetric quantum mechanics
  A. Comtet & C. Texier (1997)
- Sinai diffusion (classical)
  Bouchaud, **Comtet**, Georges & Le Doussal (1990)
- some spin chain models
  Fisher, Le Doussal, Monthus, ...

$$\langle m(x)\rangle = 0 \qquad \Rightarrow \quad \gamma_1 \underset{\varepsilon\to0}{\simeq} \frac{g}{\ln(g/\varepsilon)} \to 0 \qquad \text{(delocalisation)}$$

Transfer matrix formulation

$$m(x) = \sum_n w_n\,\delta(x - x_n)$$

$$\ell_n = x_{n+1} - x_n$$

Real energy $\varepsilon \in \mathbb{R}$

$$\Psi(x_{n+1}^-) = M_n \Psi(x_n^-)$$

$$M_n = \begin{pmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{pmatrix} \begin{pmatrix} e^{w_n} & 0 \\ 0 & e^{-w_n} \end{pmatrix} \qquad \text{where } \theta_n = \varepsilon\,\ell_n$$

$\varepsilon \in \mathrm{i}\mathbb{R}$ – Transfer matrices for $(\psi, -\mathrm{i}\chi)$

$$M_n = \begin{pmatrix} \cosh\tilde\theta_n & \sinh\tilde\theta_n \\ \sinh\tilde\theta_n & \cosh\tilde\theta_n \end{pmatrix} \begin{pmatrix} e^{w_n} & 0 \\ 0 & e^{-w_n} \end{pmatrix} \qquad \text{where } \tilde\theta_n = -\mathrm{i}\varepsilon\,\ell_n$$

$\leftrightarrow$ Ising chain

Continuum limit $w_n \to 0$ & $\ell_n \to 0$

non Gaussian white noise $\longrightarrow$ Gaussian white noise with $g = \frac{\langle w_n^2 \rangle}{\langle \ell_n \rangle}$

II. A formula for the fluctuations

From random matrix product to random process

Spinor $\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$

$$\Psi(x_{N+1}^-) = \Pi_N \Psi(x_1^-) \qquad \Rightarrow \begin{cases} x & \leftrightarrow N/\rho \\ \ln|\psi(x)| & \leftrightarrow \ln||\Pi_N|| \\ \gamma_{1,2} & \leftrightarrow \rho\,\Gamma_{1,2} \end{cases}$$

$\rho = 1/\langle \ell_n \rangle$: density of "mass impurities"

Riccati variable $z(x) \stackrel{\text{def}}{=} -\varepsilon\,\chi(x)/\psi(x) = \psi'(x)/\psi(x) - m(x)$

$$\ln|\psi(x)| = \int_0^x \mathrm{d}t\,\left[z(t) + m(t)\right]$$

$$\begin{cases} \chi' = -m\,\chi + \varepsilon\,\psi \\ \psi' = m\,\psi - \varepsilon\,\chi \end{cases} \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x} z(x) = -\varepsilon^2 - z(x)^2 - 2z(x)\,m(x)$$

Generalised Lyapunov exponent

Lyapunov exponent :

$$\gamma_1 = \lim_{x\to\infty} \frac{\langle \ln|\psi(x)|\rangle}{x}$$

Generalised Lyapunov exponent :

$$\Lambda(q) = \lim_{x\to\infty} \frac{\ln\langle|\psi(x)|^q\rangle}{x} = \sum_{n=1}^{\infty} \frac{q^n}{n!}\gamma_n$$

Paladin & Vulpiani, Phys. Rep. **156** (1987)

Result

Generator

$$\mathscr{G} = 2g\, z\, \partial_z\, z\, \partial_z - (\varepsilon^2 + z^2)\partial_z$$

Result

$$\gamma_2 = g - \langle z \ln|z/\varepsilon|\rangle + \int \mathrm{d}z\mathrm{d}z'\, z\, G(z|z') \left(z' - \frac{\varepsilon^2}{z'}\right) f(z')$$

$$\mathscr{G}^\dagger f(z) = 0 \quad \text{(stationary distribution)}$$

$$\mathscr{G}^\dagger G(z|z') = f(z) - \delta(z - z')$$

Demonstration

$$\gamma_2 = \lim_{x\to\infty} \frac{\mathrm{Var}(\ln|\psi(x)|)}{x} = g + 2 \lim_{x\to\infty} \langle z(x) \int_0^x \mathrm{d}t\ [z(t)+m(t)] \rangle_c$$

Useful relation

$$2\ln|\psi(x)/\psi(x_0)| = 2\int_{x_0}^x \mathrm{d}t\ [z(t)+m(t)]$$

$$= -\ln\left|\frac{z(x)}{z(x_0)}\right| + \int_{x_0}^x \mathrm{d}t\ \left(z(t) - \frac{\varepsilon^2}{z(t)}\right)$$

Propagator $\partial_x P_x(z|z') = \mathscr{G}^\dagger P_x(z|z')$

$$\lim_{x\to\infty} \left\langle z(x) \int_{x_0}^x \mathrm{d}t\ \left(z(t) - \frac{\varepsilon^2}{z(t)}\right)\right\rangle_c$$

$$= \int \mathrm{d}z\, z \underbrace{\int_0^\infty \mathrm{d}\tau\ [P_\tau(z|z') - f(z)]}_{=G(z|z')} \left(z' - \frac{\varepsilon^2}{z'}\right) f(z')$$

III. Limiting behaviours and numerics

- High energy (SPS)

$$\gamma_2 \simeq \gamma_1 \simeq g/2$$

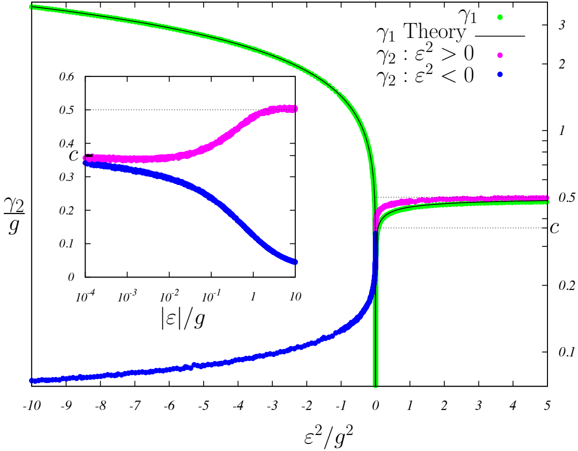
- Small energy

$$\left.\begin{array}{l} \gamma_2 \underset{\varepsilon\to 0}{\simeq} g\left[\frac{1}{3}+\frac{1}{2\ln(2g/|\varepsilon|)}\right] \\ \gamma_2 \underset{\varepsilon\to \mathrm{i}0}{\simeq} g\left[\frac{1}{3}-\frac{1}{2\ln(2g/|\varepsilon|)}\right] \end{array}\right\} \gg \gamma_1 \simeq \frac{g}{\ln(g/|\varepsilon|)}$$

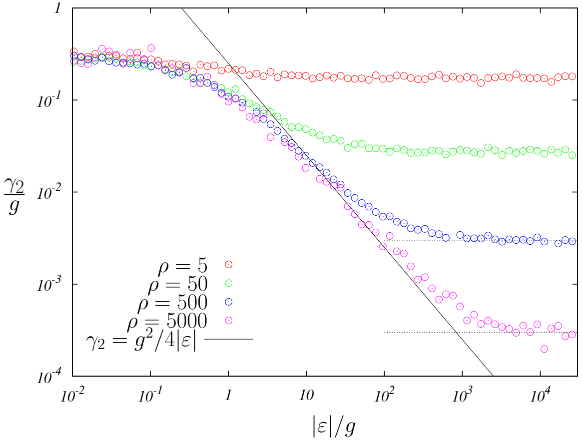
saturation of the fluctuations for $\varepsilon = 0$

- Large imaginary energy $\rightarrow$ perturbative approach for the SDE

$$\gamma_2 \underset{\varepsilon\to \mathrm{i}\infty}{\simeq} \frac{g^2}{4|\varepsilon|} \ll \gamma_1 \simeq \sqrt{|\varepsilon|}+\frac{g}{2}$$


γ_1
γ_1 Theory
$\gamma_2 : \varepsilon^2 > 0$
$\gamma_2 : \varepsilon^2 < 0$
$\frac{\gamma_2}{g}$
ε^2/g^2
$|\varepsilon|/g$
c
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5
3 2 1 0.5 0.2 0.1
0 0.1 0.2 0.3 0.4 0.5 0.6
10^{-4} 10^{-3} 10^{-2} 10^{-1} 1 10

$\varepsilon \to \mathrm{i}\infty$:



IV. Discussion

Localisation for $\varepsilon \to 0$

- Usual measure of localisation is $\widetilde{\xi}_\varepsilon = 1/\gamma_1 \simeq (1/g)\ln(g/|\varepsilon|)$ (wrong)

- Another length scale

$$\sqrt{\gamma_2 x} \gtrsim \gamma_1 x \quad \text{for} \quad x \lesssim \frac{\gamma_2}{\gamma_1^2} \sim \xi_\varepsilon = (1/g)\ln^2(g/|\varepsilon|)$$

ξ_ε : D. S. Fisher, *Random AF quantum spin chains*, PRB **50** (1994)

Localisation is dominated by fluctuations for $\varepsilon \to 0$

Relation to boundary condition sensitivity (Thouless criterion)

$$\psi(0) = \psi(L) = 0 \Rightarrow \text{spectrum } \{\varepsilon_n\}$$

$$\int_0^L \frac{\mathrm{d}x}{L} \underbrace{\langle \rho(x;\varepsilon)\rangle}_{\text{LDoS}} \simeq \overbrace{\varrho(\varepsilon)}^{\text{bulk}} \mathcal{D}(L/\xi_\varepsilon) \quad \text{where } 1/\xi_\varepsilon = \int_0^\varepsilon \mathrm{d}\varepsilon' \, \varrho(\varepsilon') \sim \frac{g}{\ln^2(g/\varepsilon)}$$

C. Texier & C. Hagendorf, J.Phys.A **43** (2010)

Merci Alain !

Appendices

Appendix A : Lyapunov exponent for $\varepsilon = 0$

The norm

$$||\Pi_N|| \stackrel{\text{def}}{=} \int_{|\Psi_0|=1} \mathrm{d}\Psi_0 \, |\Pi_N \Psi_0| \qquad \text{with } \Psi_0 = \begin{pmatrix} \sin\Theta_0 \\ -\cos\Theta_0 \end{pmatrix}$$

Case $\varepsilon = 0$

$$\Pi_N = \begin{pmatrix} e^W & 0 \\ 0 & e^{-W} \end{pmatrix} \qquad \text{with } W = \sum_{n=1}^{N} w_n$$

hence

$$|\Pi_N \Psi_0| = \sqrt{\cosh 2W - \cos 2\Theta_0 \, \sinh 2W}$$

$$|\Pi_N \Psi_0| = \sqrt{\cosh 2W - \cos 2\Theta_0 \, \sinh 2W}$$

Remark :

- $\Theta_0 = 0 \Rightarrow |\Pi_N \Psi_0| = e^{-W}$
- $\Theta_0 = \pi/2 \Rightarrow |\Pi_N \Psi_0| = e^{+W}$

Reflects the behaviour of the solutions of $\left[\sigma_2 \, \mathrm{i}\partial_x + \sigma_1 \, m(x)\right] \Psi(x) = 0$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\int^x \mathrm{d}x' \, m(x')} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\int^x \mathrm{d}x' \, m(x')}$$

The norm

$$||\Pi_N|| = \int_0^\pi \frac{\mathrm{d}\Theta_0}{\pi} \sqrt{\cosh 2W - \cos 2\Theta_0 \, \sinh 2W} = \frac{2e^{|W|}}{\pi} \, \mathbf{E}\left(\sqrt{1 - e^{-4|W|}}\right)$$

$$\ln ||\Pi_N|| \simeq |W| - \ln(\pi/2) \quad \text{for } |W| \gg 1$$

Moments

$W = \sum_n w_n \Rightarrow P_x(W) = \frac{1}{\sqrt{2\pi g x}} e^{-W^2/(2gx)}$ with $g = \rho \left\langle w_n^2 \right\rangle$, hence

$$\langle \ln ||\Pi_N|| \rangle \simeq \sqrt{\frac{2gx}{\pi}} - \ln(\pi/2)$$

$$\mathrm{Var}(\ln ||\Pi_N||) \simeq gx \left(1 - \frac{2}{\pi}\right)$$

Lyapunov

$\gamma_1 = 0$ & $\boxed{\gamma_2 = g\left(1 - \frac{2}{\pi}\right) = g \times 0.363380...}$

Appendix B : From multiplicative to additive noise

$$\frac{\mathrm{d}}{\mathrm{d}x} z(x) = -\varepsilon^2 - z(x)^2 - 2z(x)\, m(x)$$

$$\downarrow \qquad \zeta = \mp \ln(\pm z/|\varepsilon|)/2 \quad \text{for } z \in \mathbb{R}_\pm$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \zeta(x) = -\mathcal{U}'(\zeta(x)) + m(x)$$

where the potential is

$$\mathcal{U}(\zeta) = \begin{cases} \frac{|\varepsilon|}{2} \cosh 2\zeta & \text{for } \varepsilon \in \mathrm{i}\mathbb{R} \\ -\frac{|\varepsilon|}{2} \sinh 2\zeta & \text{for } \varepsilon \in \mathbb{R} \end{cases}$$

Generator

$$\mathscr{G}^\dagger = \frac{g}{2}\,\partial_\zeta^2 + \partial_\zeta \mathcal{U}'(\zeta)$$

Generalised Lyapunov exponent $\Lambda(q) = q\,\gamma_1 + \frac{q^2}{2}\,\gamma_2 + \cdots$

$$\gamma_1 = 2\,\langle \mathcal{U}(\zeta) \rangle$$
$$\gamma_2 = g - 2\,\langle \zeta\,\mathcal{U}'(\zeta) \rangle + 8 \int \mathrm{d}\zeta \mathrm{d}\zeta'\,\mathcal{U}(\zeta)\,\mathcal{G}(\zeta|\zeta')\,\mathcal{U}(\zeta')\,\mathcal{P}(\zeta')$$

where

$$\mathscr{G}^\dagger \mathcal{P}(\zeta) = 0$$
$$\mathscr{G}^\dagger \mathcal{G}(\zeta|\zeta') = \mathcal{P}(\zeta) - \delta(\zeta - \zeta')$$

$$\boxed{-\varphi_{n+1} + V_n\,\varphi_n - \varphi_{n-1} = \varepsilon\,\varphi_n}$$

Weak disorder (perturbative) expansion breaks down at band center :

- Weak disorder expansion :

$$\gamma_1 \simeq \frac{\langle V_n^2 \rangle}{4\sqrt{4-\varepsilon^2}} \overset{\varepsilon\to 0}{\longrightarrow} \frac{1}{8}\,\langle V_n^2 \rangle$$

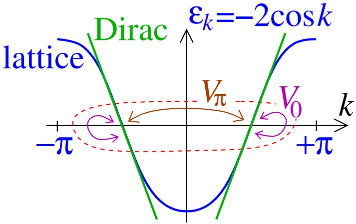
Derrida & Gardner (1984) ; J.-M. Luck, the book, (1992)

- Band center anomaly of the Lyapunov

$$\gamma_1 = \underbrace{[\Gamma(3/4)/\Gamma(1/4)]^2}_{\simeq\, 0.114}\,\langle V_n^2 \rangle \quad \text{at } \varepsilon = 0$$

Kappus & Wegner, Z. Phys. **45** (1981)
Derrida & Gardner, J. Physique **45** (1984)

Continuum limit of the lattice model



Continuum limit for $\varepsilon \simeq 0$:

$$[-\mathrm{i}\sigma_3\,\partial_x + V_0(x) + \sigma_1\,V_\pi(x)]\,\widetilde{\Psi}(x) = \varepsilon\,\widetilde{\Psi}(x)$$

- Forward scattering $V_0 \longrightarrow$ strength g_0
- Backward scattering $V_\pi \longrightarrow$ strength g

Anomaly

$$\text{deviation from SPS } \frac{\gamma_2}{\gamma_1} \xrightarrow{\text{disorder}\to 0} 1$$

Transfer matrices

Choose

$$V_0(x) = \sum_n v_n \, \delta(x - x_n) \text{ and } V_\pi(x) = \sum_n w_n \, \delta(x - x_n)$$

$$M_n = \begin{pmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{pmatrix} \begin{pmatrix} e^{w_n} & 0 \\ 0 & e^{-w_n} \end{pmatrix}$$

where $\theta_n = \varepsilon \, (x_{n+1} - x_n) - v_n$

Lyapunov in the continuum limit $\varepsilon = 0$, $v_n \to 0$ & $w_n \to 0$

$\to$ Comtet, Luck, Texier & Tourigny, J.Stat.Phys. (2013) ; § 6

$$\gamma_1 = g \left[\frac{1}{k^2} \left(\frac{\mathbf{E}(k)}{\mathbf{K}(k)} - 1 \right) + 1 \right] \quad \text{with } k = \frac{1}{\sqrt{1 + g_0/g}}$$

Check : $(g_0 = g)$

$$\gamma_1 = g \left[2 \frac{\mathbf{E}(1/\sqrt{2})}{\mathbf{K}(1/\sqrt{2})} - 1 \right] = g \left(\frac{2\Gamma(3/4)}{\Gamma(1/4)} \right)^2$$

Tune the BC anomaly at $\varepsilon = 0$

- $g_0 = g$:

$$\frac{\gamma_2}{\gamma_1} \simeq g \times 1.047$$

Schomerus & Titov, PRB **67** (2003)

- $g_0 \ll g$:

$$\frac{\gamma_2}{\gamma_1} \sim \begin{cases} \ln(g/g_0) & \text{at } \varepsilon = 0 \\ \ln(g/|\varepsilon|) & \text{for } g_0 \ll |\varepsilon| \ll g \end{cases}$$

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\sum_n v_n\,\delta(x-x_n)\right]\psi(x)=E\,\psi(x)$$

Transfer matrices

$$M_n=\begin{pmatrix}\cos\theta_n & -\sin\theta_n\\ \sin\theta_n & \cos\theta_n\end{pmatrix}\begin{pmatrix}1 & u_n\\ 0 & 1\end{pmatrix}\text{ for }E=k^2$$

$$M_n=\begin{pmatrix}\cosh\theta_n & \sinh\theta_n\\ \sinh\theta_n & \cosh\theta_n\end{pmatrix}\begin{pmatrix}1 & u_n\\ 0 & 1\end{pmatrix}\text{ for }E=-k^2$$

$$\theta_n=k(x_{n+1}-x_n)\text{ and }u_n=v_n/k$$

Riccati $z = \psi'/\psi$

$$\frac{\mathrm{d}z(x)}{\mathrm{d}x} = -E - z(x)^2 + V(x)$$

Continuum limit ($\theta_n \to 0$ and $u_n \to 0$) :

$V(x)$ a Gaussian white noise : $\langle V(x)V(x')\rangle = \sigma\,\delta(x-x')$

$$\Rightarrow \text{ generator } \mathscr{G} = (\sigma/2)\partial_z^2 - (E + z^2)\partial_z$$

Lyapunov

$$\gamma_1 = \langle z \rangle = \int \mathrm{d}z\, z\, f(z)$$

$$\gamma_2 = 2 \int \mathrm{d}z\mathrm{d}z'\, z\, G(z|z')\, z'\, f(z')$$

where

$$\mathscr{G}^\dagger f(z) = 0$$

$$\mathscr{G}^\dagger G(z|z') = f(z) - \delta(z-z')$$

Effective potential $\mathcal{U}(z) = Ez + z^3/3$

$$\mathscr{G} = (\sigma/2)\partial_z^2 - \mathcal{U}'(z)\partial_z \qquad \& \qquad \mathscr{G}^\dagger = (\sigma/2)\partial_z^2 + \partial_z \mathcal{U}'(z)$$

Stationary distribution $\mathscr{G}^\dagger f(z) = 0$

$$f(z) = \frac{2N}{\sigma} f_0(z) \int_{-\infty}^{z} \frac{\mathrm{d}t}{f_0(t)} \qquad \text{with } f_0(z) = e^{-\frac{2}{\sigma}\mathcal{U}(z)}$$

Solution of $\mathscr{G}^\dagger G(z|z') = f(z) - \delta(z - z')$

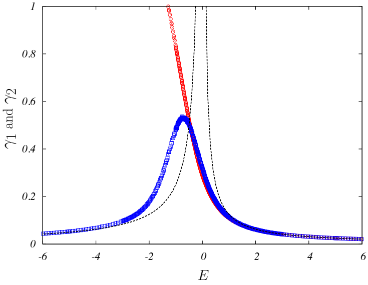
$$G(z|z') = \frac{1}{N}\left\{ f(z)\left[c(z') + \int_{-\infty}^{z} \mathrm{d}t\, f(t)\right] - f_0(z)\int_{-\infty}^{z} \mathrm{d}t\, \frac{f(t)^2}{f_0(t)} + \frac{f_0(z_>)f(z_<)}{f_0(z')} \right\}$$

$$c(z') + \frac{1}{2} = \frac{\sigma}{2N}\left[\int_{-\infty}^{+\infty} \mathrm{d}z''\, f(z'')^2 f(-z'') - f(-z')\, f(z')\right] - \int_{-\infty}^{z'} \mathrm{d}z''\, f(z'')$$

Limiting values

$$\gamma_2 \underset{E\to\infty}{\simeq} \frac{\sigma}{8E} \simeq \gamma_1 \text{ at leading order (SPS)}$$

$$\gamma_2 \underset{E\to-\infty}{\simeq} \frac{\sigma}{4(-E)} \ll \gamma_1 \simeq \sqrt{-E}$$



Generalised Lyapunov

$$\Lambda(q) = \lim_{x\to\infty} \frac{\ln \langle |\psi(x)|^q \rangle}{x} = \sum_{n=1}^{\infty} \frac{q^n}{n!} \gamma_n$$

Paladin & Vulpiani, Phys. Rep. **156** (1987)

$$\langle |\psi(x)|^q \rangle = \left\langle e^{q \int_0^x \mathrm{d}t\, z(t)} \right\rangle = \int \mathrm{d}z \, \langle\, z \,| e^{x\left(\mathscr{G}^\dagger + qz\right)} |\, z_0 \,\rangle$$

$$\underset{x\to\infty}{\sim} e^{x\Lambda(q)}$$

where

$$\left[\mathscr{G}^\dagger + q\,z\right] \Phi_0^{\mathrm{R}}(z;q) = \Lambda(q)\, \Phi_0^{\mathrm{R}}(z;q)$$

Perturbative analysis of $[\mathscr{G}^\dagger + q\,z]\Phi_0^{\mathrm{R}}(z;q) = \Lambda(q)\,\Phi_0^{\mathrm{R}}(z;q)$

Schomerus & Titov, PRE **66** (2002)

$$\Lambda(q) = q\,\gamma_1 + \frac{q^2}{2!}\,\gamma_2 + \cdots$$

$$\Phi_0^{\mathrm{R}}(z;q) = f(z) + q\,\varphi_1(z) + q^2\,\varphi_2(z) + \cdots$$

We get

$$\mathscr{G}^\dagger\varphi_1(z) = (\gamma_1 - z)\,f(z) \qquad \xrightarrow{\int \mathrm{d}z} 0 = \gamma_1 - \int \mathrm{d}z\,z\,f(z)$$

$$\mathscr{G}^\dagger\varphi_2(z) = (\gamma_1 - z)\,\varphi_1(z) + \frac{1}{2}\,\gamma_2\,f(z)$$

$$\vdots$$

We deduce Schomerus & Titov's result

$$\gamma_2 = 2 \int \mathrm{d}z\,(z-\gamma_1)\,\varphi_1(z)$$

$$\varphi_1(z) = N\left(\frac{2}{\sigma}\right)^2 f_0(z)\int_{-\infty}^{z}\frac{\mathrm{d}z'}{f_0(z')}\int_{-\infty}^{z'}\mathrm{d}z''\,(\gamma_1 - z'')\,f_0(z'')\int_{-\infty}^{z''}\frac{\mathrm{d}z'''}{f_0(z''')}$$

where $f_0(z) = e^{-\frac{2}{\sigma}\mathcal{U}(z)}$

This is a different integral representation from

$$\gamma_2 = 2\int \mathrm{d}z\mathrm{d}z'\,z\,G(z|z')\,z'\,f(z')$$