# Biaised and unbiased two-dimensional random walks and the Hofstadter model


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some results for the algebraic area of biased random walks

their relation to ”Hofstadter” quantum mechanics

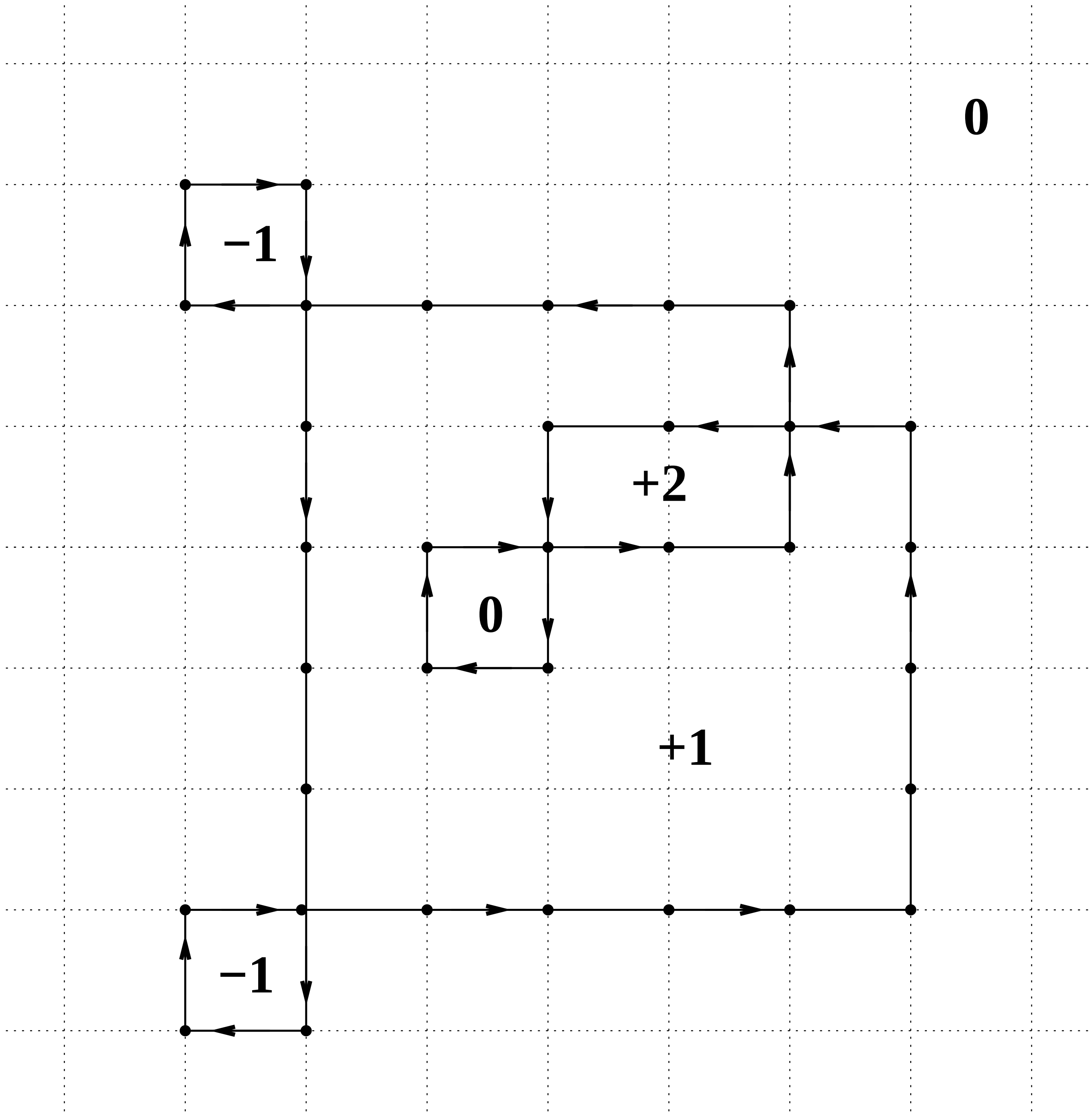
$\rightarrow$ Trace identities

$\rightarrow$ Combinatorics

**algebraic area of a closed random walk on a square lattice**

**defined in terms of its *n*-Winding Sectors**

**= points enclosed *n* times by the walk**


0
−1
+2
0
+1
−1

$S_n \equiv$ area of the $n$-winding sectors inside the walk

$$\Rightarrow \text{ algebraic area } A = \sum_{n=-\infty}^{\infty} n S_n$$

in the example above :

$$A = -1 \times 1 + -1 \times 1 + 0 \times 1 + 1 \times 21 + 2 \times 2 = 23$$

## question :

consider all closed random walks starting from and returning to a given point after $N$ steps

$$\langle A\rangle = \sum_{n=-\infty}^{\infty} n\langle S_n\rangle = 0 \quad \text{obvious}$$

## what is the algebraic area probability distribution $P_N(A)$ ?

a difficult problem

in the continuous limit $N \to \infty$, lattice spacing $a \to 0 \Rightarrow$ Brownian curves "time" $t = Na^2$

## $P_N(A) \to P_t(A)$ is known (easy) = Levy's law

mapping on a quantum particle in a $\perp$ $B$ field

$\Rightarrow P_t(A) =$ Fourier transform of Landau partition function $Z_{\text{Landau}}(B)$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} dB \; \frac{Z_{\text{Landau}}(B)}{Z_{\text{Landau}}(0)} \; e^{-iBA} = P_t(A)$$

$$\int_{-\infty}^{+\infty} dA \;\; P_t(A)\; e^{iBA} = \frac{Z_{\text{Landau}}(B)}{Z_{\text{Landau}}(0)}$$

for the case of discrete random walks on a lattice :

mapping on a quantum particle on a lattice in a $\perp$ $B$ field

$\equiv$ Hofstadter Hamiltonian

$$H_\gamma = T_x + T_x^{-1} + T_y + T_y^{-1}$$

$\gamma \equiv 2\pi\Phi/\Phi_0$ flux per unit cell

$$T_x = e^{i(p_x - eA_x)/\hbar} \qquad T_y = e^{i(p_y - eA_y)/\hbar}$$

$$T_x T_y = e^{-i\gamma} T_y T_x$$

define $C_N(A) \equiv$ number of closed random walks of $N$ steps with area $A \Rightarrow P_N(A) = \frac{C_N(A)}{\sum_A C_N(A)}$

$$\text{continuous} \ \int_{-\infty}^{+\infty} dA \ P_t(A) \, e^{iBA} = \frac{Z_{\text{Landau}}(B)}{Z_{\text{Landau}}(0)} \rightarrow \text{discrete} \sum_{A=-\infty}^{A=\infty} C_N(A) \, e^{i\gamma A} = \text{Trace } H_\gamma^N$$

replace $e^{i\gamma} \rightarrow \mathrm{Q}$

$\sum_{A=-\infty}^{A=\infty} C_N(A) \, e^{i\gamma A} \rightarrow \sum_A C_N(A) \mathrm{Q}^A \equiv Z_N(\mathrm{Q}) =$ generating function for the $C_N(A)$'s

$$Z_N(\mathrm{Q} = e^{i\gamma}) = \text{Trace } \ H_\gamma^N$$

**Trace identity (Bellissard (1997))**

little is known exactly on the $C_N(A)$'s or on Trace $H_\gamma^N$

a trivial case : $\gamma = 0 \Leftrightarrow \mathrm{Q} = 1 \rightarrow Z_N(\mathrm{Q}) = \sum_A C_N(A)\mathrm{Q}^A|_{\mathrm{Q}=1} = \sum_A C_N(A)$

$\Leftrightarrow$ $N$-steps closed random walks counting :

$N$ steps = $M$ steps right/left $\oplus (N-2M)/2$ steps up/down

$$Z_N(\mathrm{Q}=1) = \sum_{M=0}^{N/2} \frac{N!}{M!^2(\frac{N-2M}{2})!^2} = (\frac{1}{2\pi})^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (2\cos k_x + 2\cos k_y)^N dk_x dk_y$$

$2\cos k_x + 2\cos k_y$ is the spectrum of the Hofstadter Hamiltonian when $\gamma = 0$

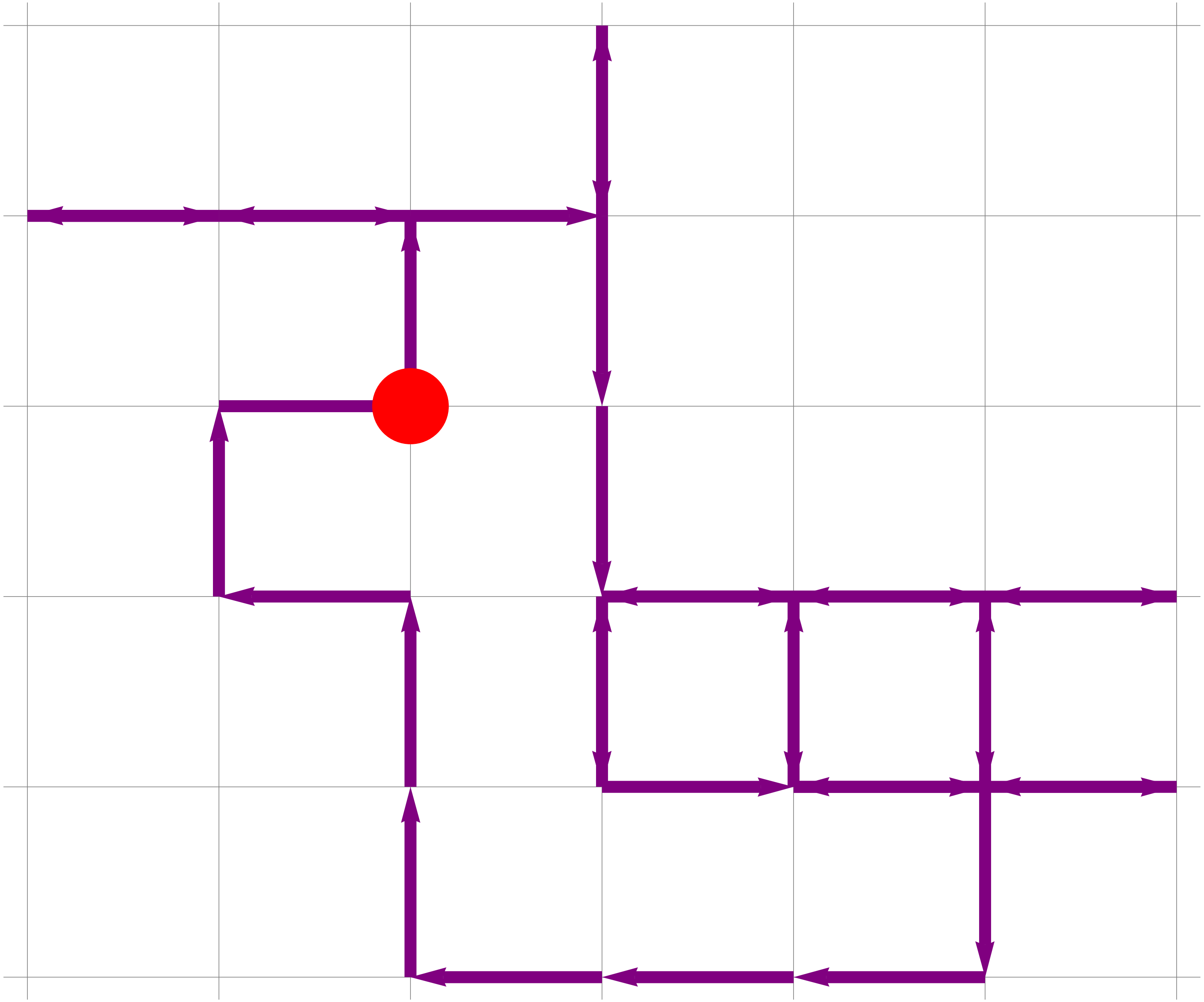
$$H_0 = T_x + T_x^{-1} + T_y + T_y^{-1}$$

$$T_x = e^{ip_x/\hbar} \quad T_y = e^{ip_y/\hbar}$$

Bloch eigenstates : $e^{ik_x x}e^{ik_y y}$ $\qquad k_x, k_y \in [-\pi, \pi]$

eigenenergies : $e^{ik_x} + e^{-ik_x} + e^{ik_y} + e^{-ik_y} = 2\cos k_x + 2\cos k_y$

$$\rightarrow Z_N(\mathrm{Q}=1) = \mathrm{Trace} H_0^N$$



## a solvable case:

## random walks biased to go only to the right

## never to the left

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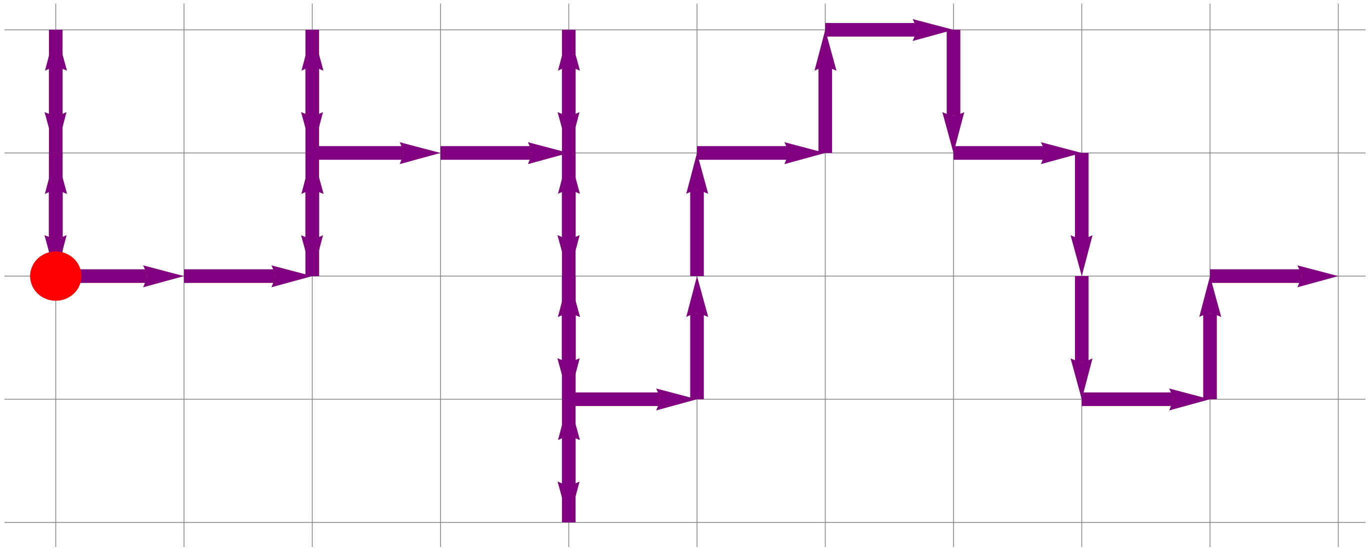


$M$ steps right $\oplus L_1$ steps up $\oplus L_2$ steps down

one knows the generating function $Z_{M,L_1,L_2}(\mathrm{Q})$ for the $C_{M,L_1,L_2}(A)$'s

S. Mashkevich, S.O. (2009)

$$Z_{M,L_1,L_2}(\mathrm{Q}) = \sum_{k=0}^{\min(L_1,L_2)} \left[ \binom{M+L_1+L_2}{k} - \binom{M+L_1+L_2}{k-1} \right] \binom{M+L_1-k}{M}_{\mathrm{Q}^{-1}} \binom{M+L_2-k}{M}_{\mathrm{Q}}$$

involves Q-binomial, Q-factorials

$$\binom{N}{M}_{\mathrm{Q}} \equiv \frac{[N]_{\mathrm{Q}}!}{[M]_{\mathrm{Q}}![N-M]_{\mathrm{Q}}!}$$

$$[N]_{\mathrm{Q}}! = \prod_{i=1}^{N} \frac{1-\mathrm{Q}^i}{1-\mathrm{Q}} = 1(1+\mathrm{Q})(1+\mathrm{Q}+\mathrm{Q}^2)\cdots(1+\mathrm{Q}+\ldots+\mathrm{Q}^{N-1})$$

$(1+\mathrm{Q}+\ldots+\mathrm{Q}^{N-1}) =$ Q-deformation of the integer $N$

algebraic area $\Leftrightarrow$ non commuting space $\Leftrightarrow$ Q-deformation we are looking at a Trace identity which would be analogous to

$$Z_N(\mathrm{Q} = e^{i\gamma}) = \text{Trace } H_\gamma^N$$

$\Rightarrow$ we still need "closed" walks after $N = M + L_1 + L_2$ steps

$\Rightarrow$ starting point and ending point on the same horizontal axis : $L_1 = L_2$

$N = M + L_1 + L_2 \Rightarrow L_1 = L_2 = (N - M)/2$

$N$ steps = $M$ steps right $\oplus (N - M)/2$ steps up $\oplus (N - M)/2$ steps down

still one has to identify the starting point and the ending point :

boundary conditions (see later)

on which quantum model are mapped these biased ”closed” random walks ?

one way : look again at $\mathrm{Q} = 1 \Leftrightarrow$ random walks counting

$Z_{M,L_1,L_2}(\mathrm{Q}=1) = \frac{(M+L_1+L_2)!}{M!L_1!L_2!}$

total number of $N$-steps biased eandom walks :

$$\sum_{M=0}^{N} Z_{M,\frac{N-M}{2},\frac{N-M}{2}}(\mathrm{Q}=1) = \sum_{M=0}^{N} \frac{N!}{M!(\frac{N-M}{2})!^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pm 1 + 2\cos k_y)^N dk_y$$

$\rightarrow$ spectrum

$$\pm 1 + 2\cos k_y$$

corresponds to quantum Hamiltonian with only right hoppings on the horizontal axis

$$H_0 = T_x + T_y + T_y^{-1}$$

eigenstates

$$e^{ik_x x} e^{ik_y y}$$

eigenenergies

$$e^{ik_x} + e^{ik_y} + e^{-ik_y} = e^{ik_x} + 2\cos k_y$$

if one restricts Hilbert space to real spectrum then $k_x = 0, \pm\pi$

$$\pm 1 + 2\cos k_y$$

$$k_x = \text{boundary conditions} \qquad k_y = \text{quantum number}$$

take now $\mathrm{Q} \neq 1 \Leftrightarrow \gamma \neq 0$

$\rightarrow$ mapping on non Hermitian "Hofstadter" model

$$H_\gamma = T_x + T_y + T_y^{-1}$$
$$T_x T_y = e^{-i\gamma} T_y T_x$$

$\rightarrow$ Trace identity ?

$$\text{"}\sum_M\text{"}\ Z_{M,\frac{N-M}{2},\frac{N-M}{2}}(\mathrm{Q} = e^{i\gamma}) = \text{Trace}\ \ H_\gamma^N$$

on the quantum mechanics side :

non Hermitian Hamiltonian is solvable

$\rightarrow$ spectrum is known in the commensurate case $\gamma = 2\pi \frac{p}{q}$

in fact $\gamma = \frac{2\pi}{q}$ (spectrum does not depend on $p$)

for a given $q$ : eigenstates $E_q(r)$

$$E_q(r) = 2\cos\left[\frac{\arccos[e^{iqk_x}/2 + \cos(qk_y)]}{q} + 2\pi\frac{1}{q}r\right] \qquad r = 1, 2, \ldots, q$$

complex eigenenergies for certain $k_y$ (non Hermitian)

Trace is

$$\text{Trace } H^N_{\gamma=2\pi/q} \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} dk_y \frac{1}{q}\sum_{r=1}^{q} E_q(r)^N$$

Trace identity :

use quantum lattice is $q$-periodic on the $x$-axis (up to phase $e^{iqk_x}$)

$\rightarrow$ $q$-periodic sum over $M$ :

one verifies (Mathematica) S. Matveenko, S.O. (2013)

$$\sum_{M=0,q,2q,\ldots}^{N} Z_{M,\frac{N-M}{2},\frac{N-M}{2}}(\mathrm{Q}=e^{i2\pi/q})e^{iMk_x} = \text{Trace } H^N_{\gamma=2\pi/q}$$

what have we learned from quantum mechanics :

i) "$q$-periodic" walks $M = 0, q, 2q, \ldots =$ multiple of $q$

ii) generating function evaluated at roots of unity $\mathrm{Q} = e^{i2\pi/q}$

we notice that in the Trace identity each term

$$Z_{M,\frac{N-M}{2},\frac{N-M}{2}}(\mathrm{Q} = e^{i2\pi/q})$$

with $M = 0, q, 2q, \ldots$ is an integer

why integers show up and what is their combinatorial meaning (if any) ?

S. Mashkevich, S.O., A. Polychronakos (2014) why integers ?

due to well-known Q-binomial identity : when $M$ multiple of $q$

$$\binom{M+L}{M}_{\mathrm{Q}=e^{\frac{2i\pi p}{q}}} = \binom{\left[\frac{M+L}{q}\right]}{\frac{M}{q}} = \text{ an integer}$$

$\Rightarrow$ when $\mathrm{Q} = e^{\frac{2i\pi}{q}}$ the $(M, L_1, L_2)$ walks algebraic area generating function

$$Z_{M,L_1,L_2}(\mathrm{Q}) = \sum_{k=0}^{\min(L_1,L_2)} \left[\binom{M+L_1+L_2}{k} - \binom{M+L_1+L_2}{k-1}\right] \binom{M+L_1-k}{M}_{\mathrm{Q}^{-1}} \binom{M+L_2-k}{M}_{\mathrm{Q}}$$

$=$ an integer when $M$ multiple of $q$

$M$ multiple of $q \Rightarrow$ simplify further by taking also $L_1 - L_2$ multiple of $q$

in particular when $q = 2 \to \mathrm{Q} = -1$

$$Z_{M,L_1,L_2}(\mathrm{Q} = -1) = \binom{L_1 + L_2}{L_1} \binom{\frac{M+L_1+L_2}{2}}{\frac{L_1+L_2}{2}}$$

even more generally for unbiaised $(M_1, M_2, L_1, L_2)$ walks

take both $M_1 - M_2$ and $L_1 - L_2$ multiple of $q$

again when $q = 2 \to \mathrm{Q} = -1$

$$Z_{M_1,M_2,L_1,L_2}(\mathrm{Q} = -1) = \binom{L_1 + L_2}{L_1} \binom{M_1 + M_2}{M_1} \binom{\frac{M_1+M_2+L_1+L_2}{2}}{\frac{L_1+L_2}{2}}$$

simple expressions : combinatorial meaning ?

$q = 2 \rightarrow \mathrm{Q} = -1$ is well-known in combinatorics

as the $\mathrm{Q} = -1$ Stembridge phenomenon

particular case of the more general $\mathrm{Q} =$ root of unity Sieving phenomenon

Sieving paramount example :

i) collection of the $L$-subsets of the $(M+L)$-set $= \{1,2,\ldots,M+L\}$

ii) generating function $\binom{M+L}{L}_{\mathrm{Q}} = \mathrm{Q}$-binomial

iii) $c =$ cycling generator acting by cycling $1 \to 2, 2 \to 3, \ldots, M+L \to 1$

Sieving : $\binom{M+L}{L}_{\mathrm{Q}}$ evaluated at $\mathrm{Q} = \exp(2i\pi p/(M+L))$ counts the number of the $L$-subsets which are fixed by $c^p$

example : $M=2$, $L=2$ one has $\binom{2+2}{2}_{\mathrm{Q}} = 1+\mathrm{Q}+2\mathrm{Q}^2+\mathrm{Q}^3+\mathrm{Q}^4$

$p = 2 \to \mathrm{Q} = \exp(2i\pi 2/(2+2)) = -1 \leftrightarrow$ Stembridge

$1+\mathrm{Q}+2\mathrm{Q}^2+\mathrm{Q}^3+\mathrm{Q}^4 = 2$

2 is the number of the 2-subsets of $\{1,2,3,4\}$ fixed by $c^2$ : $\{1,3\} \leftrightarrow \{2,4\}$

$p = 1,3 \Rightarrow 1+\mathrm{Q}+2\mathrm{Q}^2+\mathrm{Q}^3+\mathrm{Q}^4 = 0 \Rightarrow$ no 2-subset fixed by $c$ or $c^3$

$p = 4 \to \mathrm{Q} = 1 \Rightarrow = 6 \Rightarrow$ all 2-subsets fixed by $c^4 = 1$ (trivial) one to one mapping

the collection of the $L$-subsets of the $(M+L)$-set

$$\leftrightarrow$$

the collection of $(M,L)$ random walks = $M$-steps right $\oplus$ $L$-steps up

for example :

$\{1,3\}$ subset of $\{1,2,3,4\}$ $\leftrightarrow$ Right-Up-Right-Up $(M=2, L=2)$ walk

and not surprisingly the algebraic area generating function for $(M,L)$ walks

$$Z_{M,L_1=L,L_2=0}(\mathrm{Q}) = \binom{M+L}{L}_{\mathrm{Q}}$$

indeed reduces to the Q-binomial Sieving generating function

$\rightarrow$ Sieving interpretation of

$$Z_{M_1,M_2,L_1,L_2}(\mathrm{Q}=-1) = \binom{L_1+L_2}{L_1}\binom{M_1+M_2}{M_1}\binom{\frac{M_1+M_2+L_1+L_2}{2}}{\frac{L_1+L_2}{2}}$$

in terms of subsets of sets

a daunting question : $Z_{M_1,M_2,L_1,L_2}(\mathrm{Q})$ seems out of reach

still when taking both $M_1 - M_2$ and $L_1 - L_2$ multiple of $q$ what could be the generalization of the above formula to

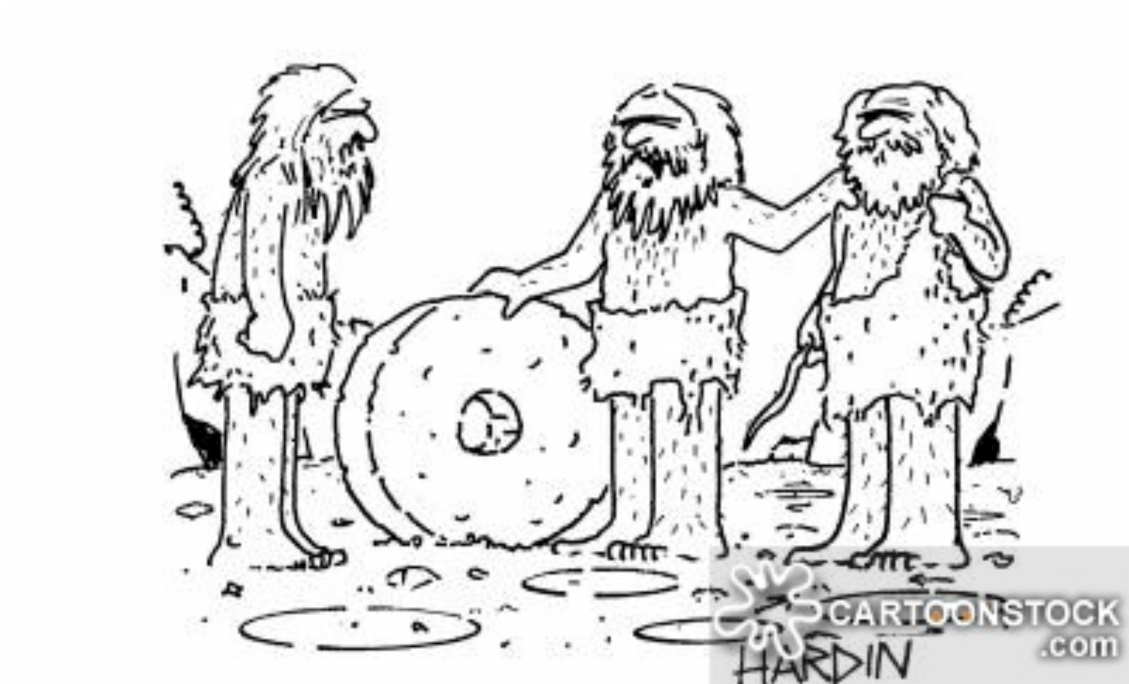
$$Z_{M_1,M_2,L_1,L_2}(\mathrm{Q} = \mathrm{e}^{\frac{2\mathrm{i}\pi}{q}})$$

$\rightarrow$ sum rule for the Hofstadter spectrum : $Z_N(\mathrm{Q} = e^{2\mathrm{i}\pi/q}) = \text{Trace } H^N_{\gamma=2\pi/q}$

$$Z_N(\mathrm{Q}) = \sum_{M=0}^{N/2} Z_{M,M,N/2-M,N/2-M}(\mathrm{Q})$$

Bonne continuation Alain

et merci pour tout !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