On the Area Law for Disordered Quasifree Fermions

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Entanglement Entropy
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Quasifree Fermions
Area Law for Disordered Fermions
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Entanglement: a complex and delicate quantum phenomenon (since 1930’ Einstein et al, Schrodinger), in particular a widely believed resource of quantum informatics (since 1980’ Feynman).

Consider a bipartite quantum system consisting of two parts A(lice) and B(ob), thus with the state space

\[ \mathcal{H}_{A+B} = \mathcal{H}_A \otimes \mathcal{H}_B. \]

Pure state \( \Psi \in \mathcal{H}_{A+B} \) is entangled if it is not separable

\[ \Psi = \Psi_A \otimes \Psi_B, \quad \Psi_{A,B} \in \mathcal{H}_{A,B}. \]

Example: Bell states. A and B are qubits, i.e., dim \( \mathcal{H}_{A,B} = 2 \), \( (|1\rangle_A, |2\rangle_A) \) and \( (|1\rangle_B, |2\rangle_B) \) are orthonormal bases and

\[ \Psi = 2^{-1/2} (|1\rangle_A \otimes |2\rangle_B \pm |2\rangle_A \otimes |1\rangle_B). \]
**Entanglement Entropy (Generalities)**

**Entanglement Entropy**

**Entropy** (i) classical physics: a measure of the lack of knowledge (e.g., on microstates corresponding a given macrostate), hence related to classical probability or randomness

(ii) quantum physics: a measure of quantum correlations due to the "randomness" of quantum mechanics. One uses the **von Neumann** entropy of a state $\rho$

$$S(\rho) = - \text{Tr} \rho \log_2 \rho.$$  

Renyi entropy: $R_\alpha(\rho) = -(\alpha - 1)^{-1} \log_2 \text{Tr} \rho^\alpha$ and \(\lim_{\alpha \to 1} R_a(\rho) = S(\rho)\) ("replica" version)

An important property: if $\dim \mathcal{H} = n$, then

$$\max_\rho S(\rho) = \log_2 n$$

(as for classical entropy).
If $\rho$ is a density matrix of a bipartite system $A + B$, then (Dirac) 

$$\rho_A = \text{Tr}_B \rho$$

is the reduced density matrix of $A$. 

**Entanglement entropy** of $A$: 

$$S(\rho_A) = \text{Tr}_A \rho_A \log_2 \rho_A.$$ 

Example: If $\Psi$ is the Bell state, then $S(|\Psi\rangle \langle \Psi|) = 0$ (valid for any pure state). However, 

$$\rho_A = 2^{-1}(|1\rangle_A \langle 1|_A + |2\rangle_A \langle 2|_A)$$

and $S(\rho_A) = \log_2 2 = 1$, i.e., is maximal possible, i.e., $S(\rho_A)$ is an entanglement quantifier.
Let $A + B$ be a macroscopic bipartite system in the $d$-dimensional volume $\Omega$ of linear size $L$, $A$ be its part in the $d$-dimensional volume $\Lambda$ of linear size $l$, and $B = \Omega \setminus \Lambda$ (environment). One is interested in the asymptotics of $S(\rho_\Lambda)$ for $1 \ll l \ll L$.

Recall the large distance behavior of binary (ternary, etc.), just take $A = \{x, y\}$ and let $l = |x - y| \to \infty$. Important in the analysis of pt’s.

**Difference:** $S_\Lambda$ is highly non-local, hence the qpt order parameter?.

According to Bekenstein, 1973, Hawking, 1974 (black holes physics); Bombelli et al., 1986, Srednicki, 1993 (QFT), Callan-Wilczek, 1994 (CFT); Calabrese-Cardy, 2005th (CFT, Quantum Spin Chains)

$$S(\rho_\Lambda) \simeq \begin{cases} 
\text{no qpt,} & l^{d-1}, \\
\text{qpt,} & l^{d-1} \log l, \\
\text{area law,} & \\
\text{violation of area law.} & 
\end{cases}$$

Recently: The area law is valid "generically" for locally interacting quantum systems having a gap in their spectrum (*Hastings 2010th*).
The violation of the area law:

(i) $d = 1$: explicitly solvable models, e.g., 1d quantum spin chains.

(ii) $d > 1$: mostly conjectured, established only for toy model of quasi-free translation invariant fermions, i.e., quadratic Hamiltonians

$$\hat{H} = \sum_{j,k \in \Omega} D_{jk} c_j^+ c_k + \frac{1}{2} \sum_{j,k \in \Omega} O_{jk} c_j^+ c_k^+ + \frac{1}{2} \sum_{j,k \in \Omega} \overline{O}_{kj} c_j c_k$$

where $D = D^*$, $O^T = -O$, $\overline{O} = \{\overline{O}_{jk}\}_{j,k=1}^n$.

Consider (for simplicity) the "diagonal" case $O = 0$. Denote

$$K = \{< c_j c_k^+ >\}_{j,k \in \Omega}, \quad K^0 = K \mid_{T=0}, \quad K_\Lambda = K^{(0)} \mid_{\Lambda}$$

Then

$$S(\rho_\Lambda) = -\text{Tr} \rho_\Lambda \log_2 \rho_\Lambda = \text{tr} h(K_\Lambda),$$

$$h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x), \quad 0 \leq x \leq 1,$$

where $\text{Tr}$ and $\text{tr}$ denote the trace in the $2^{|\Omega|}$ - and $|\Omega|$-dimensional spaces.
Nice formulas but not too simple to use even in the translation invariant case.

For the quadratic Hamiltonian with finite range and translation invariant $D$ the large-$l$ scaling of the entanglement entropy for any $d \geq 1$ (both critical $l^{d-1} \log l$ and non critical $l^{d-1}$) was established

(i) via upper and lower bounds,

(ii) via certain conjectures on the subleading term in the Szego theorem for Toeplitz determinants (Gioev-Klich 06, Wolf 08)

(iii) rigorously, by using a rather sophisticated techniques of modern operator theory and harmonic analysis (Sobolev et al 14).

It turns out that the disordered case is in a way simpler (modulo basic results of localization theory)
Choose $D = (H - E_F)/T$, where $H$ is the Hamiltonian of the $d$-dimensional Anderson model with random i.i.d. potential $V = \{V_j\}_{j \in \Omega}$ and $E_F$ is the Fermi energy lying in the bulk of spectrum of $H$. Then

$$K^0 = K \bigg|_{T=0} = \theta(E_F - H),$$

where $\theta$ is the Heaviside function.

Thus, $K^0$ is the orthogonal projection on the ground state of the whole system (the Slater determinant on the first $n$ eigenstates of $\hat{H}$, where $n/|\Omega| = N(E_F)$ and $N(E_F)$ is the Integrated Density of States of $H$ and the entropy of the whole system is zero.

Start from the bounds (simple calculus or simple Maple)

$$\varphi(x) \leq h(x) \leq (\varphi(x))^a, \quad \varphi(x) = 4x(1 - x), \quad 0 \leq a \leq \ln 2.$$
If $K^0 = \{P_{jk}\}_{j,k \in \mathbb{Z}^d}$, then $K_\Lambda = \{P_{jk}\}_{j,k \in \Lambda}$ and for $a = 1/2$ in the upper bound

$$L_\Lambda \leq S_\Lambda \leq U_\Lambda, \quad L_\Lambda = 4 \text{tr} \Gamma_\Lambda, \quad U_\Lambda = 2 \text{tr} \sqrt{\Gamma_\Lambda}, \quad \Gamma_\Lambda = K_\Lambda (\mathbf{1}_\Lambda - K_\Lambda).$$

Denote $\overline{\Lambda}$ the exterior of $\Lambda$ and use the equality $\sum_{k \in \mathbb{Z}^d} |P_{jk}|^2 = P_{jj}$:

$$(\Gamma_\Lambda)_{jk} = \sum_{t \in \overline{\Lambda}} P_{jt} P_{tk}, \quad j, k \in \Lambda.$$ 

The large-$\Lambda$ behavior of $\Gamma_\Lambda$ is determined by the large $|j - k|$ decay of $P_{jk}$. $\Gamma_\Lambda$ can be expressed via the current-current correlator determining the a.c. conductivity. It is also closely related to the number statistics in $\Lambda$. 
In the 1d translation invariant case

\[ P_{jk} = \frac{\sin p_F |t|}{|t|}, \]

\[ S_\Lambda \gtrsim 8 \sum_{t=1}^{\infty} t \Pi_t, \quad \Pi_{j-k} = |P_{jk}|^2 \]

and we have

\[ S_\Lambda \gtrsim \left(\frac{4}{\pi^2}\right) \log l, \quad l >> 1, \]

i.e., the violation of the area law (in the 1d case the boundedness of \( S_\Lambda \)).

Likewise, for \( d \geq 1 \):

\[ S_\Lambda \gtrsim l^{d-1} \log l, \quad l >> 1. \]
Mean Entanglement Entropy

A fundamental result on the Anderson localization is the bound

$$\langle |P_{jk}| \rangle \leq Ce^{-\gamma |j-k|}$$

valid for a translation invariant in mean and short correlated random potentials and

(i) $1d$ case: all energies and and strengths of disorder;
(ii) $d \geq 2$ case: neighborhoods of band edges (any disorder) and for all energies if the disorder is large enough.

Since $|P_{jk}| \leq 1$, we have for $\Pi_{j-k} = \langle |P_{jk}|^2 \rangle \leq Ce^{-\gamma |j-k|}$ and the above lower bound implies

$$\langle S_\Lambda \rangle \gtrsim c_- l^{d-1}, \ l \gg 1!$$

This suggests the validity of the area law scaling for the mean entropy and any $d$ if $E_F$ is in the localized spectrum.
Mean Entanglement Entropy
Upper Bound

It follows from the above that

\[ \langle S_\Lambda \rangle \leq \langle U_\Lambda \rangle, \quad U_\Lambda = 2 \text{tr} \sqrt{\Gamma_\Lambda}, \quad \Gamma_\Lambda = K_\Lambda (1_\Lambda - K_\Lambda). \]

Use the Peierls inequality: for any convex \( f \) \( (f'' \leq 0) \) and hermitian \( M \)

\[ \text{tr} \ f(M) \leq \sum_j f(M_{jj}) \]

with \( f(x) = 2\sqrt{x} \) and \( M = \Gamma_\Lambda \) and some calculus to obtain

\[ \langle U_\Lambda \rangle \lesssim 4(2^d - 1)l^{d-1} \sum_{j=0}^{\infty} \left( \sum_{k=1}^{\infty} \Pi_{k+j} \right)^{1/2}, \]

and

\[ c_- l^{d-1} \leq \langle S_\Lambda \rangle \leq c_+ l^{d-1}, \quad 0 \leq c_- \leq c_+ < \infty. \]

We have the area law scaling for the mean entanglement entropy and
\( d \leq 1 \) if the fermi energy is in the localized spectrum.
Write $\Lambda = [-m, m]$, $l = 2m + 1$ and obtain $L_m \leq S_m \leq U_m$, where for $m \gg 1$

$$L_m \simeq \mathcal{L}^+_0(T^m V) + \mathcal{L}^-_0(T^{-m} V),$$

the shift operator $T$ act on $V = \{V_j\}_{j=-\infty}^{\infty}$ as $(TV)_j = V_{j+1}$ and, e.g.,

$$\mathcal{L}^+_0 = 4 \sum_{j=-\infty}^{0} \sum_{k=1}^{\infty} |P_{jk}|^2$$

The double sum is not zero and finite for all typical realizations (with probability 1) again because of the exponential localization bound. Likewise

$$U_m \simeq \mathcal{U}^+_0(T^m V) + \mathcal{U}^-_0(T^{-m} V).$$
Bounds for the Entropy of Typical Realizations (1d case)

Histograms
Histograms overlap, hence the entanglement entropy depends nontrivially on the realizations of disorder, i.e., *is not selfaveraging* for $m \gg 1$. Indeed, if the entropy were nonrandom, then the whole probability distribution of the upper bound has to lie on the right of that of lower bound.
Bounds for the Entropy of Typical Realizations (1d case)

Conclusions

- Histograms overlap, hence the entanglement entropy depends nontrivially on the realizations of disorder, i.e., *is not selfaveraging* for $m \gg 1$. Indeed, if the entropy were nonrandom, then the whole probability distribution of the upper bound has to lie on the right of that of lower bound.

- Histograms are independent of $l = 2m + 1 \gtrsim 15000$. Indeed, if the random potential is short correlated, the terms of the both bounds are statistically independent for $m \gg 1$, and since the potential is translation and reflection symmetric in the mean, the probability distributions of these terms are identical. Hence, for $m \gg 1$ the probability distribution of the r.h.s. of both bounds are the convolutions of those of $L_0^\pm$ and $U_0^\pm$. This is also confirmed by our numerics.
Convolutions: Lower Bound

Histograms

\[ \rho(t) \]

- \( L_m \)
- \( \text{conv } L_0^- \)
- \( \text{conv } L_0^+ \)
Convolutions: Upper Bound

Histograms
Entropy is bounded with probability 1, i.e., satisfies the *stochastic area law*, if its distribution is concentrated on a finite interval. Otherwise, the entropy has to have "peaks" $s_n \to \infty$, $n \to \infty$, where $s_n$ solves $p(s_n) \approx n^{-(1+\delta)}$, $\delta > 0$ with $p(s)$ the large-$s$ tail of the entropy probability distribution. In particular, if $p(s) \approx e^{-s/s_0}$, then $s_n \approx s_0(1 + \delta) \log n$, corresponding to the critical scaling of the entropy. Note, however, that $s_n$'s are just extremal and rather rare peaks of randomly fluctuating entropy but not its "regular" asymptotics.
Emergence of the Area Law

Weak Disorder

\[ y = P_1 \log(x) + P_2 \]

\begin{align*}
P_1 & = 0.22096 \pm 0.00173 \\
P_2 & = -0.17898 \pm 0.00645 
\end{align*}