On the Area Law for Disordered Quasifree Fermions

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Outline

- Entanglement Entropy
- Area Law for Translation Invariant Systems
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Entanglement: a complex and delicate quantum phenomenon (since 1930' *Einstein et al, Schrodinger*), in particular a widely believed resource of quantum informatics (since 1980' *Feynman*).

Consider a **bipartite** quantum system consisting of two parts A(lice) and B(ob), thus with the state space

$$\mathcal{H}_{A+B} = \mathcal{H}_A \otimes \mathcal{H}_B .$$

Pure state $\Psi \in \mathcal{H}_{A+B}$ is **entangled** if it is not **separable**

$$\Psi = \Psi_A \otimes \Psi_B, \; \Psi_{A,B} \in \mathcal{H}_{A,B}.$$

Example: **Bell states**. A and B are qubits, i.e., $\dim \mathcal{H}_{A,B} = 2$, $(|1\rangle_A, |2\rangle_A)$ and $(|1\rangle_B, |2\rangle_B)$ are orthonormal bases and

$$\Psi = 2^{-1/2} \left(|1\rangle_A \otimes |2\rangle_B \pm |2\rangle_A \otimes |1\rangle_B \right)$$

Entropy (i) classical physics: a measure of the lack of knowledge (e.g., on microstates corresponding a given macrostate), hence related to classical probability or randomness
(ii) quantum physics: a measure of quantum correlations due to the "randomness" of quantum mechanics. One uses the **von Neumann** entropy of a state ρ

$$S(\rho) = -\mathrm{Tr}\rho \log_2 \rho.$$

Renyi entropy: $R_\alpha(\rho) = -(\alpha-1)^{-1}\log_2 \mathrm{Tr}\rho^\alpha$ and $\lim_{\alpha\to 1} R_a(\rho) = S(\rho)$ ("replica" version)

An important property: if $\dim \mathcal{H} = n$, then

$$\max_\rho S(\rho) = \log_2 n$$

(as for classical entropy).

Entanglement Entropy (Generalities)

Reduced Density Matrix

If ρ is a density matrix of a bipartite system $A+B$, then (*Dirac*)

$$\rho_A = \mathrm{Tr}_B \rho$$

is the **reduced density matrix** of A.
Entanglement entropy of A:

$$S(\rho_A) = \mathrm{Tr}_A \ \rho_A \log_2 \rho_A .$$

Example: If Ψ is the Bell state, then $S(|\Psi\rangle \langle\Psi|) = 0$ (valid for any pure state). However,

$$\rho_A = 2^{-1}(|1\rangle_A \langle 1|_A + |2\rangle_A \langle 2|_A)$$

and $S(\rho_A) = \log_2 2 = 1$, i.e., is maximal possible, i.e., $S(\rho_A)$ is an **entanglement quantifier**.

Let $A + B$ be a macroscopic bipatite system in the d-dimensional volume Ω of linear size L, A be its part in the d-dimensional volume Λ of linear size l, and $B = \Omega \setminus \Lambda$ (environment). One is interested in the asymptotics of $S(\rho_\Lambda)$ for $1 << l << L$.

Recall the large distance behavior of binary (ternary, etc.), just take $A = \{x, y\}$ and let $l = |x - y| \to \infty$. Important in the analysis of pt 's.

Difference: S_Λ is highly non-local, hence the qpt order parameter?.

According to *Bekenstein, 1973, Hawking, 1974* (black holes physics); *Bombelliet al., 1986, Srednicki, 1993* (QFT), *Callan-Wilczek*, 1994 (CFT); *Calabrese-Cardy*, 2005th (CFT, Quantum Spin Chains)

$$S(\rho_\Lambda) \simeq \begin{cases} \textit{no qpt}, & l^{d-1}, \qquad \textit{area law}, \\ \textit{qpt}, & l^{d-1}\log l, \quad \textit{violation of area law}. \end{cases}$$

Recently: The area law is valid "generically" for locally interacting quantum systems having a gap in their spectrum (*Hastings 2010th*).

The violation of the area law:
(i) $d = 1$: explicitly solvable models, e.g., 1d quantum spin chains.
(ii) $d > 1$: mostly conjectured, established only for toy model of quasi-free translation invariant fermions, i.e., quadratic Hamiltonians

$$\widehat{H} = \sum_{j,k \in \Omega} D_{jk} c_j^+ c_k + \frac{1}{2} \sum_{j,k \in \Omega} O_{jk} c_j^+ c_k^+ + \frac{1}{2} \sum_{j,k \in \Omega} \overline{O}_{kj} c_j c_k$$

where $D = D^*$, $O^T = -O$, $\overline{O} = \{\overline{O}_{jk}\}_{j,k=1}^n$.
Consider (for simplicity) the "diagonal" case $O = 0$. Denote

$$K = \{< c_j c_k^+ >\}_{j,k \in \Omega},\ K^0 = K\,|_{T=0}\,,\ K_\Lambda = K^{(0)}\,|_\Lambda$$

Then

$$\begin{aligned} S(\rho_\Lambda) &= -\mathrm{Tr}\ \rho_\Lambda \log_2 \rho_\Lambda = \mathrm{tr}\ h(K_\Lambda), \\ h(x) &= -x \log_2 x - (1-x) \log_2 (1-x),\ 0 \le x \le 1, \end{aligned}$$

where Tr and tr denote the trace in the $2^{|\Omega|}$ - and $|\Omega|$-dimensional spaces.

Nice formulas but not too simple to use even in the translation invariant case.

For the quadratic Hamiltonian with finite range and translation invariant D the large-l scaling of the entanglement entropy for any $d \geq 1$ (both critical $l^{d-1} \log l$ and non critical l^{d-1}) was established
(i) via upper and lower bounds,
(ii) via certain conjectures on the subleading term in the Szego theorem for Toeplitz determinants (*Gioev-Klich 06, Wolf 08*)
(iii) rigorously, by using a rather sophisticated techniques of modern operator theory and harmonic analysis (*Sobolev et al 14*).

It turns out that the disordered case is in a way simpler (modulo basic results of localization theory)

Choose $D = (H - E_F)/T$, where H is the Hamiltonian of the d-dimensional Anderson model with random i.i.d. potential $V = \{V_j\}_{j \in \Omega}$) and E_F is the Fermi energy lying in the bulk of spectrum of H. Then

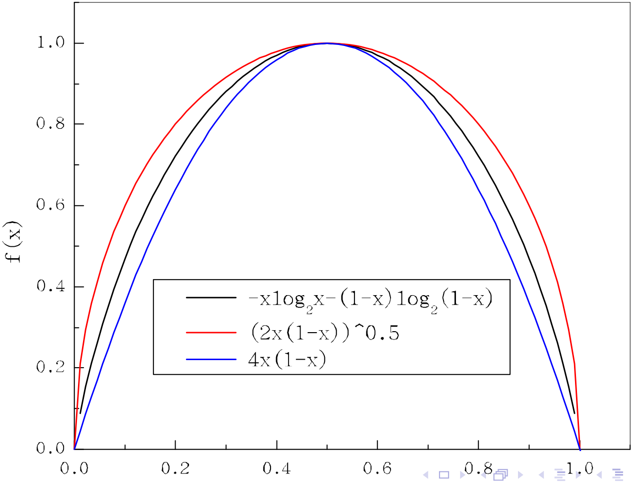
$$K^0 = K\,|_{T=0} = \theta(E_F - H),$$

where θ is the Heaviside function.

Thus, K^0 is the orthogonal projection on the ground state of the whole system (the Slater determinant on the first n eigenstates of $\widehat{H}$, where $n/|\Omega| = N(E_F)$ and $N(E_F)$ is the Integrated Density of States of) H and the entropy of the whole system is zero.
Start from the bounds (simple calculus or simple Maple)

$$\varphi(x) \leq h(x) \leq (\varphi(x))^a,\ \varphi(x) = 4x(1-x),\ 0 \leq a \leq \ln 2.$$

Bounds for Entanglement Entropy



If $K^0 = \{P_{jk}\}_{j,k \in \mathbb{Z}^d}$, then $K_\Lambda = \{P_{jk}\}_{j,k \in \Lambda}$ and for $a = 1/2$ in the upper bound

$$L_\Lambda \le S_\Lambda \le U_\Lambda,\ L_\Lambda = 4\ \mathrm{tr}\Gamma_\Lambda,\ U_\Lambda = 2\ \mathrm{tr}\sqrt{\Gamma_\Lambda},\ \Gamma_\Lambda = K_\Lambda(\mathbf{1}_\Lambda - K_\Lambda).$$

Denote $\overline{\Lambda}$ the exterior of Λ and use the equality $\sum_{k \in \mathbb{Z}^d} |P_{jk}|^2 = P_{jj}$:

$$(\Gamma_\Lambda)_{jk} = \sum_{t \in \overline{\Lambda}} P_{jt} P_{tk},\ j,k \in \Lambda.$$

The large-Λ behavior of Γ_Λ is determined by the large $|j-k|$ decay of P_{jk}.

Γ_Λ can be expressed via the current-current correlator determining the a.c. conductivity. It is also closely related to the number statistics in Λ.

In the 1d translation invariant case

$$P_{jk} = \frac{\sin p_F |t|}{|t|},$$

$$S_\Lambda \gtrsim 8 \sum_{t=1}^{\infty} t\Pi_t, \ \Pi_{j-k} = |P_{jk}|^2$$

and we have

$$S_\Lambda \gtrsim (4/\pi^2) \log l, \ l >> 1,$$

i.e., the violation of the area law (in the 1d case the boundedness of S_Λ). Likewise, for $d \geq 1$:

$$S_\Lambda \gtrsim l^{d-1} \log l, \ l >> 1.$$

A fundamental result on the Anderson localization is the bound

$$\langle |P_{jk}| \rangle \leq Ce^{-\gamma||j-k|}$$

valid for a translation invariant in mean and short correlated random potentials and
(i) $1d$ case: all energies and and strengths of disorder;
(ii) $d \geq 2$ case : neighborhoods of band edges (any disorder) and for all energies if the disorder is large enough.
Since $|P_{jk}| \leq 1$, we have for $\Pi_{j-k} = \langle |P_{jk}|^2 \rangle \leq Ce^{-\gamma|j-k|}$ and the above lower bound implies

$$\langle S_\Lambda \rangle \gtrsim c_- l^{d-1}, \; l >> 1!$$

This suggests the validity of the area law scaling for the mean entropy and any d if E_F is in the localized spectrum.

It follows from the above that

$$\langle S_\Lambda \rangle \leq \langle U_\Lambda \rangle\,,\ U_\Lambda = 2\,\mathrm{tr}\sqrt{\Gamma_\Lambda},\ \Gamma_\Lambda = K_\Lambda(\mathbf{1}_\Lambda - K_\Lambda).$$

Use the Peierls inequality: for any convex f ($f'' \leq 0$) and hermitian M

$$\mathrm{tr}\ f(M) \leq \sum_j f(M_{jj})$$

with $f(x) = 2\sqrt{x}$ and $M = \Gamma_\Lambda$ and some calculus to obtain

$$\langle U_\Lambda \rangle \lesssim 4(2^d - 1)l^{d-1} \sum_{j=0}^{\infty} \left(\sum_{k=1}^{\infty} \Pi_{k+j} \right)^{1/2},$$

and

$$c_- l^{d-1} \leq \langle S_\Lambda \rangle \leq c_+ l^{d-1}, \quad 0 \leq c_- \leq c_+ < \infty.$$

We have the area law scaling for the mean entanglement entropy and $d \leq 1$ if the fermi energy is in the localized spectrum.

Write $\Lambda = [-m, m]$, $l = 2m + 1$ and obtain $L_m \leq S_m \leq U_m$, where for $m >> 1$

$$L_m \simeq \mathcal{L}_0^+(T^m V) + \mathcal{L}_0^-(T^{-m} V),$$

the shift operator T act on $V = \{V_j\}_{j=-\infty}^{\infty}$ as $(TV)_j = V_{j+1}$ and, e.g.,

$$\mathcal{L}_0^+ = 4 \sum_{j=-\infty}^{0} \sum_{k=1}^{\infty} |P_{jk}|^2$$

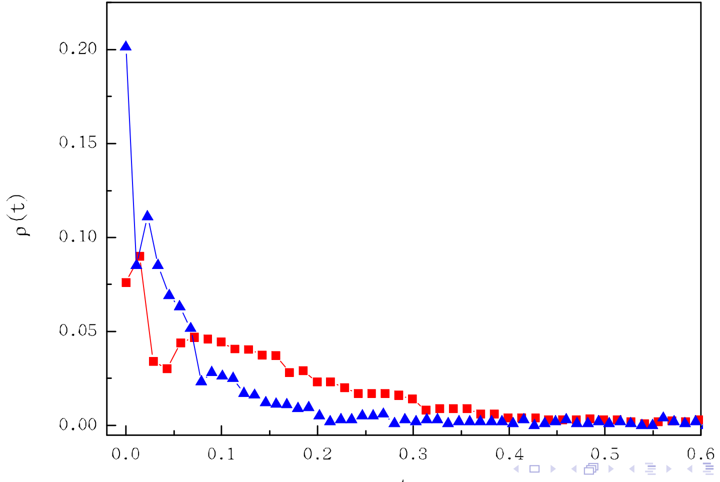
The double sum is not zero and finite for all typical realizations (with probability 1) again because of the exponential localization bound.

Likewise

$$U_m \simeq \mathcal{U}_0^+(T^m V) + \mathcal{U}_0^-(T^{-m} V).$$

Bounds for the Entropy of Typical Realizations (1d case)

Histograms



- Histograms overlap, hence the entanglement entropy depends nontrivially on the realizations of disorder, i.e., *is not selfaveraging* for $m >> 1$. Indeed, if the entropy were nonrandom, then the whole probability distribution of the upper bound has to lie on the right of that of lower bound.

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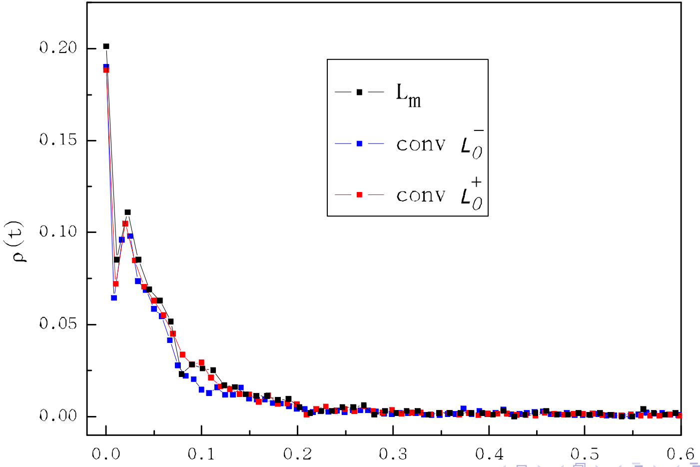
- Histograms are independent of $l = 2m + 1 \gtrsim 15000$. Indeed, if the random potential is short correlated, the terms of the both bounds are statistically independent for $m >> 1$, and since the potential is translation and reflection symmetric in the mean, the probability distributions of these terms are identical. Hence, for $m >> 1$ the probability distribution of the r.h.s. of both bounds are the convolutions of those of $\mathcal{L}_0^{\pm}$ and $\mathcal{U}_0^{\pm}$. This is also confirmed by our numerics.

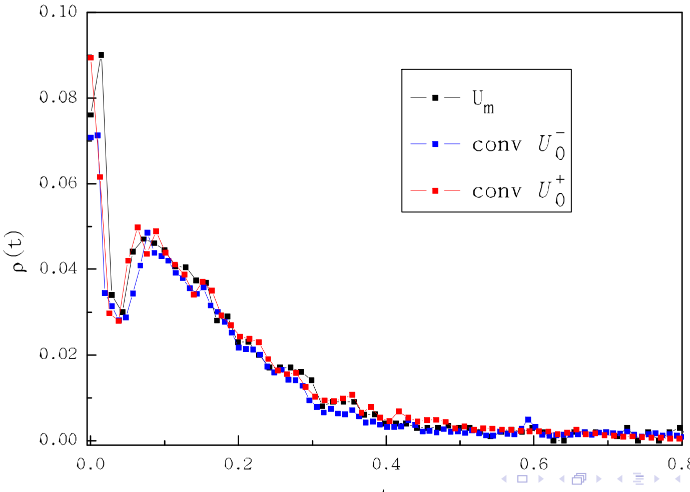
Convolutions: Lower Bound

Histograms



Convolutions: Upper Bound

Histograms

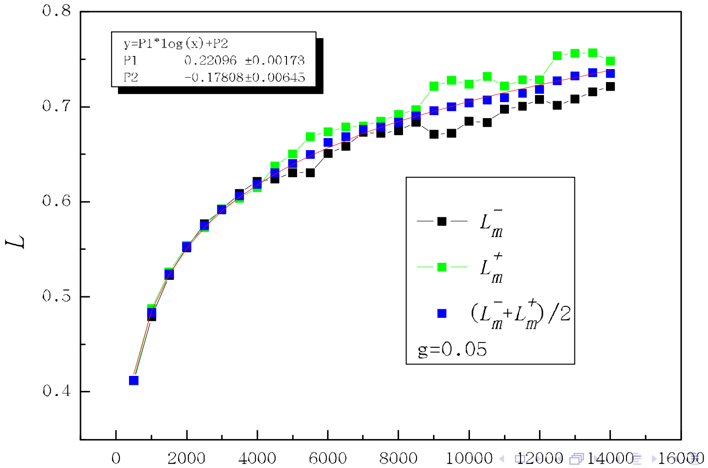


More Conclusions

- Entropy is bounded with probability 1, i.e., satisfies the *stochastic area law*, if its distribution is concentrated on a finite interval. Otherwise, the entropy has to have "peaks" $s_n \to \infty$, $n \to \infty$, where s_n solves $p(s_n) \simeq n^{-(1+\delta)}$, $\delta > 0$ with $p(s)$ the large-s tail of the entropy probability distribution. In particular, if $p(s) \simeq e^{-s/s_0}$, then $s_n \simeq s_0(1+\delta)\log n$, corresponding to the critical scaling of the entropy. Note, however, that s_n's are just extremal and rather rare peaks of randomly fluctuating entropy but not its "regular" asymptotics.

Emergence of the Area Law

Weak Disorder



Emergence of the Area Law

Stronger Disorder