A Short Walk along Quantum Trajectories, in the Company of Alain

(On jumps and spikes in strong continuous measurement)

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Plan

1. Alain (and I)
2. Open quantum systems and partial measurements
3. On jumps and spikes

Alain (and I)

Alain, my teacher

- In 1986-1987
  - Coherent states in QFT
  - Kählen-Lehmann representation
  - BRS transformation
  - Landau levels
  - Ward identities in QED
  - One-loop effective potential for Φ^4
  - Path integrals to prove θ-function identities
  - ...

Alain, an inspirational colleague

- During the period 1990-2014 we met at most a few times a year. But each encounter was a great occasion for me to learn.
  - The local time
  - The Ray-Knight theorems
  - The local time and Bessel processes
  - ...

Alain in administration

- During the period 1990-2014 we met at most a few times a year.
  - His role in the creation of LPTMS
  - His role at IHP
  - His role as a member of the IPhT evaluation committee
  - His role for the connexions between probabilists and theoretical physicists (... well, this is not just administration)
  - ...
  - His views about the recent changes in the French science landscape

I hope you understand my deep gratitude for Alain, and why I'm so proud to be here today.

Open quantum systems and partial measurements

Open quantum systems

- Open quantum systems are ubiquituous in nature
  - Transport phenemena
  - Contacts with reservoirs
  - ...
- Nonunitary evolution of a part from unitary evolution of the whole Universe
- The body of knowledge is huge ...
- ... but today's aims are very modest :
  - Markovian setting
  - Non-unitary evolution due to partial measurements
  - An example of quantum trajectories with
    - unexpected
    - but universal

    features

Open quantum systems

- The Hilbert space of the "Universe" splits as :

$$\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$$

- the state of $\mathcal{H}$ is described by a density matrix ρ
- The "one time step" evolution operator on $\mathcal{H}$ is U

$$\rho \to U \rho U^\dagger$$

- The subsystem described by $\mathcal{H}_s$ is our real interest ...
  - The relevant information is encoded in partial traces

$$\rho_s := \mathrm{Tr}_{\mathcal{H}_e} \rho$$

Effect of U on ρ_s ? Complicated in general !

Measurement

- Measure an observable Λ on $\mathcal{H}_e$ after one time step:
  - One time step evolution

$$\rho \to U\rho U^\dagger$$

  - Measurement of Λ on $\mathcal{H}_e$

$$U\rho U^\dagger \to \frac{\langle\, i \,|\, U\rho U^\dagger \,|\, i \,\rangle \otimes |\, i \,\rangle\langle\, i \,|}{\mathrm{Tr}_{\mathcal{H}} \langle\, i \,|\, U\rho U^\dagger \,|\, i \,\rangle \otimes |\, i \,\rangle\langle\, i \,|}$$

with probability $p_i := \mathrm{Tr}_{\mathcal{H}} \langle\, i \,|\, U\rho U^\dagger \,|\, i \,\rangle \otimes |\, i \,\rangle\langle\, i \,|$.

- Basis $|\, i \,\rangle$ diagonalizes $\Lambda := \sum_i \lambda_i |\, i \,\rangle\langle\, i \,|$.

- $p_i := \mathrm{Tr}_{\mathcal{H}_s} \langle\, i \,|\, U\rho U^\dagger \,|\, i \,\rangle$.

Reminder

- If O is an operator on $\mathcal{H}$ and $|\, i \,\rangle \in \mathcal{H}_e$

$$(\mathrm{Id}_{\mathcal{H}_s} \otimes |\, i \,\rangle\langle\, i \,|) O (\mathrm{Id}_{\mathcal{H}_s} \otimes |\, i \,\rangle\langle\, i \,|) =: \langle\, i \,|\, O \,|\, i \,\rangle \otimes |\, i \,\rangle\langle\, i \,|$$

- Don't forget $\langle\, i \,|\, O \,|\, i \,\rangle$ is still an operator on $\mathcal{H}_s$.

Measurement (2)

- If $\rho = \rho_s \otimes |\,\psi\,\rangle\langle\,\psi\,|$ the effect on ρ_s is simple :

$$\rho_s \to \rho'_s := \frac{A_i \rho_s A_i^\dagger}{\mathrm{Tr}_{\mathcal{H}_s} A_i \rho_s A_i^\dagger} \text{ with proba } p_i := \mathrm{Tr}_{\mathcal{H}_s} A_i \rho_s A_i^\dagger$$

  - where $A_i := \langle\, i\,|\,U\,|\,\psi\,\rangle$ (still an operator on $\mathcal{H}_s$)

- Consistency relation

$$\sum_i A_i^\dagger A_i = \mathrm{Id}_{\mathcal{H}_s}$$

- Arbitrary consistent families A_i's can be obtained by unitary evolutions

- The map $\rho_s \to \rho'_s$ defines a random dynamical system
  - Though we cheated, iteration is meaningful
- Orbits are called Quantum Trajectories

On jumps and spikes

An illustrative example

- $\mathcal{H}_s$ has dimension $d = 2$
- $\{A_i\} = \{A_+, A_-\}$
- ρ_s, A_+, A_- are real
- $A_+^2 + A_-^2 = \mathrm{Id}_{\mathcal{H}_s}$

- $\rho_s := \frac{1}{2}(\mathrm{Id} + X\sigma_z + K\sigma_x)$
- $A_+ := \begin{pmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta \\ -\sin\alpha\sin\beta & \cos\alpha\sin\beta \end{pmatrix}$
- $A_- := \begin{pmatrix} \cos\alpha\sin\beta & \sin\alpha\sin\beta \\ -\sin\alpha\cos\beta & \cos\alpha\cos\beta \end{pmatrix}$

Purification

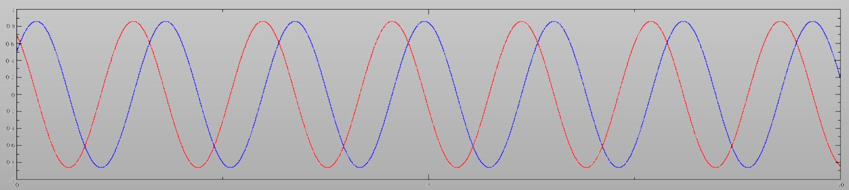
- Unless $|\sin 2\beta| = 1$
  - The iterates of $\det\rho_s$ decrease randomly but exponentially to 0, i.e. iterates of ρ_s purify
- Consequence for $d = 2$ of the general identity
  - $\mathbb{E}\left((\det\rho_s')^{\frac{1}{d}}\right) = (\det\rho_s)^{\frac{1}{d}} \sum_i (\det A_i^\dagger A_i)^{\frac{1}{d}} \leq (\det\rho_s)^{\frac{1}{d}}$
  - Supermartingale convergence theorem

An illustrative example (2)

- $\beta = \pi/4$: Hamiltonian evolution of the system

$$X' = X\cos 2\alpha + K\sin 2\alpha \quad K' = -X\sin 2\alpha + K\cos 2\alpha$$

$\rightarrow$ Rabi oscillations (α small)

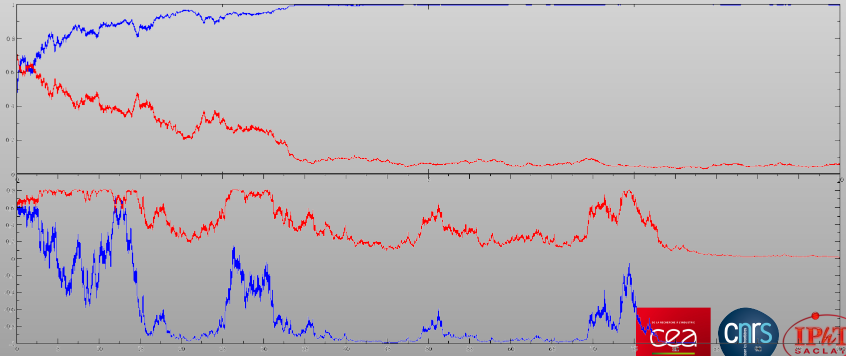


An illustrative example (3)

- $\alpha = 0$: progressive nondemolition measurement (martingales)

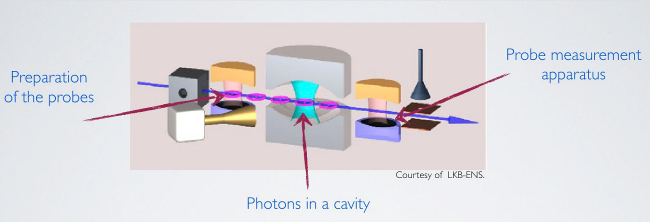
$$X' = \frac{X + \cos 2\beta}{1 + X\cos 2\beta} \qquad K' = \frac{K \sin 2\beta}{1 + X\cos 2\beta} \text{ with prob } \frac{1 + X\cos 2\beta}{2}$$

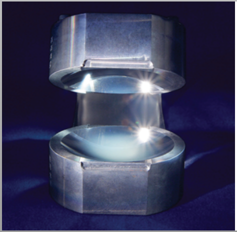
$$X' = \frac{X - \cos 2\beta}{1 - X\cos 2\beta} \qquad K' = \frac{K \sin 2\beta}{1 - X\cos 2\beta} \text{ with prob } \frac{1 - X\cos 2\beta}{2}$$

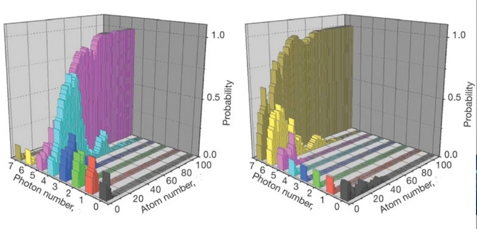


An illustrative example (3')

- Pure progressive measurement is realized in experiments :
  - C. Guerlin et al (including S. Haroche), Nature 448 (2007) 889-893, Progressive field-state collapse...

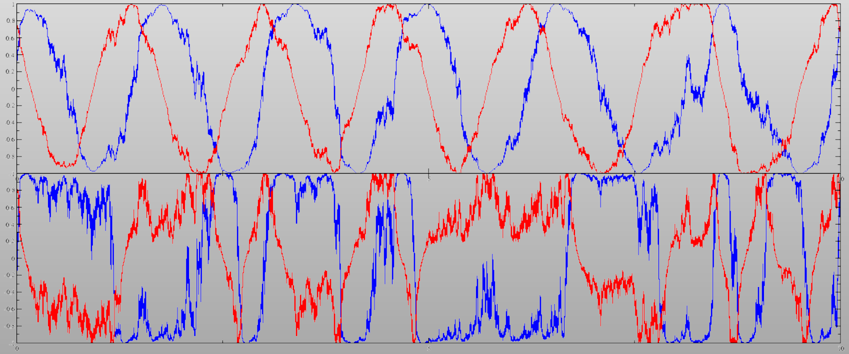






An illustrative example (4)

- What if Hamiltonian dynamics and measurement are in competition ?



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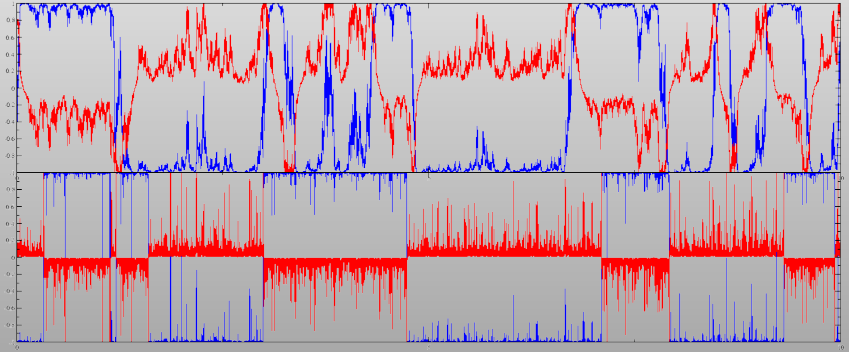

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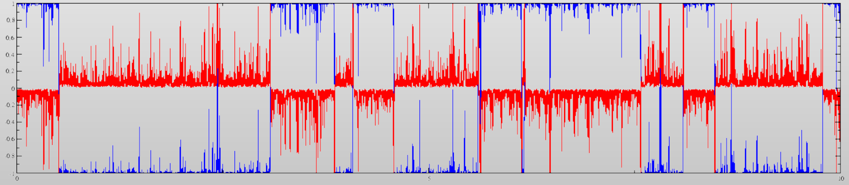
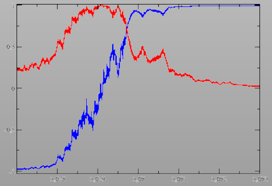
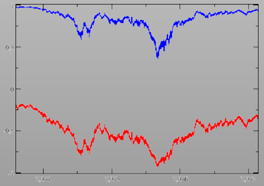
- What if Hamiltonian dynamics and measurement are in competition ?



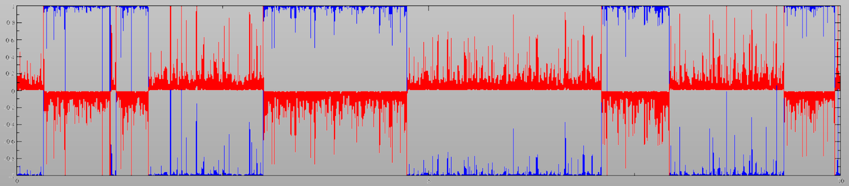
- Details of the strong measurement regime





An illustrative example (5)

- Goal : understand the features of the strong measurement regime
  - The jumps of X and their statistics
  - The sign changes of K
  - The spikes of X, K and their statistics
- ... in the continuous time limit



An illustrative example (6)

The continuous time limit

- $dX_t = \gamma u K_t dt - \gamma(1 - X_t^2) dB_t$
- $dK_t = -(\frac{\gamma^2}{2} K_t + \gamma u X_t) dt + \gamma K_t X_t dB_t$
  - $\dot{X}_t = \gamma u K_t - \gamma(1 - X_t^2)\xi_t \quad \dot{K}_t = -(\frac{\gamma^2}{2} K_t + \gamma u X_t) + \gamma K_t X_t \xi_t \quad \langle \xi_s \xi_t \rangle = \delta(t - s)$

- Purification (after a time $\sim \gamma^{-2}$, $X_t^2 + K_t^2 \simeq 1$)
  - $X_t := \cos\Theta_t \quad K_t := -\sin\Theta_t$
  - $d\Theta_t = (\gamma u - \frac{\gamma^2}{2} \sin\Theta_t \cos\Theta_t) dt + \gamma \sin\Theta_t dB_t$
- Near $\theta = k\pi$: sign changes of K

Quantum Zeno effect

- γ^{-2} is the time scale of measurement
- u is the parameter for Hamiltonian evolution
- Why rescale $u \to \gamma u$?

An illustrative example (7)

Invariant measure

- By classical formulæ:

$$\mu(\theta) = \frac{2J}{\gamma^2} \frac{e^{-\frac{2u}{\gamma}\cot\theta}}{\sin^3\theta} \int_\theta^\pi d\eta \sin\eta e^{\frac{2u}{\gamma}\cot\eta} \qquad J \text{ is the current}$$

- The tentative currentless invariant measure $\mu(\theta) \propto \sin^{-3}\theta e^{\frac{-2u}{\gamma}\cot\theta}$ is not integrable

Jump mean waiting time

- The waiting time is $T = 1/J$

$$T = \frac{2}{\gamma^2} \int_0^\pi d\theta \frac{e^{-\frac{2u}{\gamma}\cot\theta}}{\sin^3\theta} \int_\theta^\pi d\eta \sin\eta e^{\frac{2u}{\gamma}\cot\eta} \sim_{\gamma\to\infty} \frac{1}{u^2} + \cdots$$

where $\cdots$ involves horrible and inaccessible logarithms

A nice detour

Inaccessible? Look at

- *The Lyapunov Exponent of Products of Random 2x2 Matrices Close to the Identity*
  - Comtet, A., Luck, J-M., Texier, C. & Tourigny, Y.
  - Journal of Statistical Physics. 150, 1, p. 13-65.
- Nice trick: the Lyapunov Exponent has a companion, i.e is the real part of an analytic function.

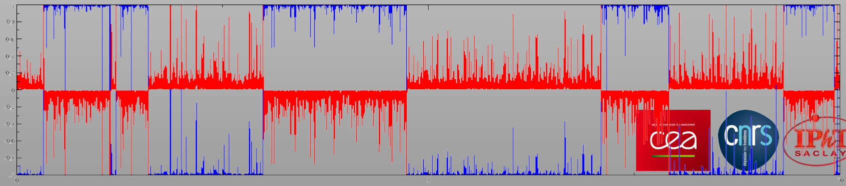
- We are indeed looking at products of random matrices close to the identity, though they are not independent
- Nevertheless, the trick allows to rewrite the double integral as a quotient of two simple integrals
- In the case at hand, a quotient of Bessel functions
  $\rightarrow$ Large γ expansion is explicit ! ! !
  - Alas (? !), the corrections play no role in the sequel

An illustrative example (8)

"Theorem" Part 1

- For large γ the finite dimensional distributions of the process $Q_t := \frac{1+X_t}{2}$ converge weakly to those of a finite state Markov process $\overline{Q}_t$ on $\{0,1\}$.
- The Markov generator is $\begin{pmatrix} -u^2 & u^2 \\ u^2 & -u^2 \end{pmatrix}$
- The $\gamma|K_t|$ (resp. $\gamma^2(1-X_t)^2$) become independent random variables with distribution $4u^2k^{-3}e^{-2u/k}dk$ (resp. $2u^2x^{-2}e^{-2u/\sqrt{x}}dx$)

- $\rho = \begin{pmatrix} Q & \cdot \\ \cdot & \cdot \end{pmatrix}$



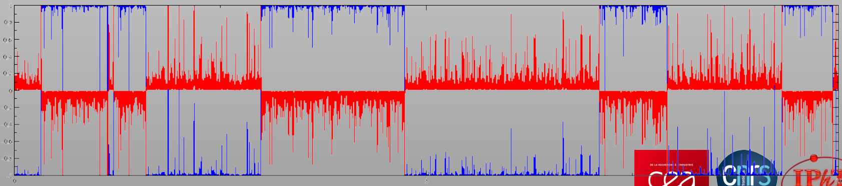
An illustrative example (8)

"Theorem" Part 2

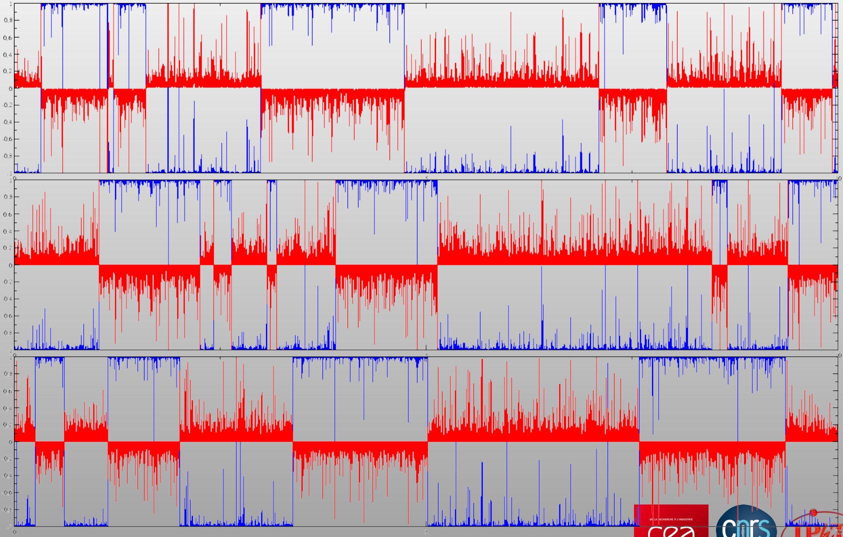
- The extrema of the spikes of Q_t converge to spacetime Poisson point processes on $[0,1] \times [0,+\infty[$ with intensity

$$\left(Q^{-2}dQ + \delta(1-Q)\right) u^2 dt \text{ or } \left((1-Q)^{-2}dQ + \delta(Q)\right) u^2 dt$$

  - Corrolary : the process Q_t does not converge weakly to the finite state process $\overline{Q}_t$.
  - Emergence of scale invariance



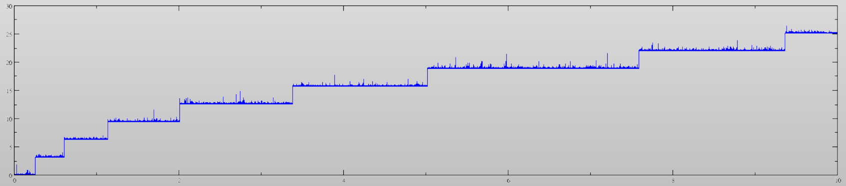
An illustrative example (9)



An illustrative example (10)

Comments

- The spikes can be seen as aborted jumps ...
- ... or the jumps as successful spikes



- It is high-jumping with a bar at height 1 and in average q^{-1} missed jumps above q before each success.

- In the large γ limit,
  - the Hamiltonian evolution turns into a boundary condition
  - in the bulk everything is dominated by measurement

Conclusions

- Quantum jumps and quantum trajectories are observed daily in experiments
- The usual interpretation is that there are intrisic jumps in the measurement randomness
- The other features (spikes ?) are interpeted as imperfections of the apparatus and/or as the effect of other sources of noise

- Spikes are a definite sign that something else is going on
- The features presented in the illustrative example :
  - emergence of jumps
  - finite state Markov process limit
  - spikes with their scale invariant distribution

  are completely universal for strong partial measurement experiments in the continuous time limit !

  $\rightarrow$ A non-standard limit theorem BBT, in preparation