Distribution of the ground state energy in a random potential

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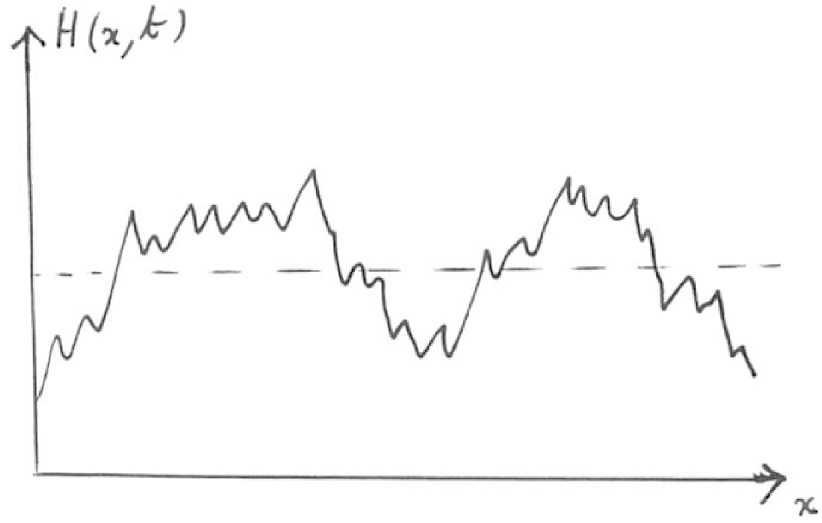
Extreme value statistics

- What is the distribution of the maximum or the minimum of a set of random variables $[x_1, x_2, \ldots, x_n]$?
- Independant identically distributed random variables
  $Z_n = \max(x_1, x_2, \ldots, x_n)$
  $P[a_n(Z_n - b_n)] \leqslant x] \to G(x)$
  $\Rightarrow$ three limiting distributions GUMBEL, FRÉCHET, WEIBULL
  $\Rightarrow$ Several applications in climatology, finance, statistical physics of disordered systems (REM).
  $\Rightarrow$ Compare with E.J. Gumbel: Les valeurs extrêmes des distributions statistiques, Annales Institut Henri Poincaré, tome 5 (1935) 115.
- Correlated random variables
  Statistical physics: source of ideas, examples and inspiration

Fluctuating Edwards Wilkinson interface

$$\frac{\partial H(x,t)}{\partial t} = \frac{\partial^2 H(x,t)}{\partial x^2} + \eta(x,t)$$

- $[H(x_1,t), H(x_2,t), \ldots, H(x_n,t)]$ is a set of correlated random variables
- Statistics of the relative height $h(x,t) := H(x,t) - \frac{1}{L}\int_0^L H(x,t)dx$
- Maximal height fluctuation $h_m(t) := \sup_{x\in[0,L]} h(x,t)$?



- Stationary regime $t \gg L^2$

$$P(h_m, L) = \frac{1}{\sqrt{L}} f\left(\frac{h_m}{\sqrt{L}}\right)$$

- Airy distribution function

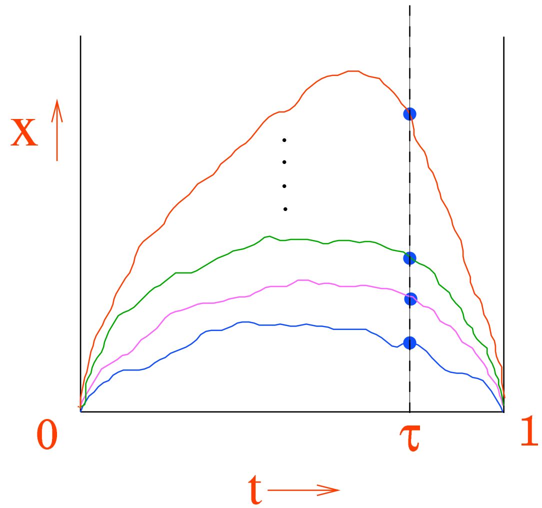
$$f(x) = \frac{2\sqrt{6}}{x^{10/3}} \sum_{k=1}^{\infty} e^{-b_k/x^2} b_k^{2/3} U(-5/6, 4/3, b_k/x^2)$$

(A.C, S. Majumdar, 2004)

- Universality (G. Schehr, S. Majumdar, 2006)

Non intersecting Brownian excursions

- interfaces → Brownian excursions: $x_1(\tau) < x_2(\tau) < .... < x_p(\tau)$



Joint law at fixed time

$$P_\tau(x_1, x_2, ..., x_p) = Z\,(\tau(1-\tau))^{-p(p+1/2)} \prod_{1 \leqslant j < k \leqslant p} (x_j^2 - x_k^2)^2$$

$$\times \prod_{i=1}^{p} x_i^2 \exp - \sum_{i=1}^{p} \frac{x_i^2}{2\tau(1-\tau)}$$

- Statistics of the top curve at fixed τ

$$2^{2/3}p^{1/6}\left(\frac{x_p(\tau)}{\sqrt{2\tau(1-\tau)}}-2\sqrt{p}\right)\to TW(\beta=2)$$

- Statistics of the bottom curve (C.A. Tracy, H. Widom, 2007)
- Disordered systems: Ground state energy (spin glasses, directed polymers) $\to$ non trivial probability laws
- Quantum systems: ground state energy of a quantum dot ?
- Toy model: One dimensional Schrödinger equation with a random potential

$$\text{Quantum process} \stackrel{\text{Riccati}}{\longrightarrow} \text{Diffusion process}$$

$$\text{Ground state energy} \stackrel{\text{Riccati}}{\longrightarrow} \text{First passage time}$$

3 models $\rightarrow$ 3 different extreme value distributions

- White noise Hamiltonian $H = -\frac{d^2}{dt^2} + V(t)$
- Supersymmetric Hamiltonian $H = Q^+ Q$
- Airy Hamiltonian $H = -\frac{d^2}{dt^2} + t + \frac{2}{\sqrt{\beta}} b'(t)$

Generalized beta ensembles

- Joint density: $P(\lambda_1, \lambda_2, ..., \lambda_n) = \frac{1}{Z} e^{-\frac{1}{2}\sum_{k=1}^{n} \lambda_k^2} \prod_{k<j} |\lambda_j - \lambda_k|^\beta$
- Matrix model realization (Trotter, Dumitriu, Edelman)

$$H_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}g_1 & \chi_{\beta(n-1)} & 0 & 0 & . \\ \chi_{\beta(n-1)} & \sqrt{2}g_2 & \chi_{\beta(n-2)} & 0 & \\ 0 & \chi_{\beta(n-2)} & \sqrt{2}g_3 & \chi_{\beta(n-3)} & \\ . & . & . & . & \chi_\beta \\ . & . & . & \chi_\beta & \sqrt{2}g_n \end{pmatrix}$$

$f_n(\chi) = \frac{1}{2^{n/2-1}\Gamma(n/2)} \chi^{n-1} e^{-\chi^2/2}, g = N(0,1)$

- Discrete Schrödinger operator
- Edge scaling $-\tilde{H}_n = -\sqrt{\frac{2}{\beta}} n^{1/6} (H_n - \sqrt{2\beta n}\, \mathbb{I})$

- Convergence to the Airy Hamiltonian
  $H = -\frac{d^2}{dt^2} + t + \frac{2}{\sqrt{\beta}} b'(t) \in L^2(R^+)$
- Distribution of the ground state energy for $\beta = 2$

$$\mathbb{P}(E_0 > E) = e^{-\int_{-E}^{\infty} (\tau+E) q^2(\tau) d\tau}$$

$$q''(\tau) = 2q^3(\tau) + \tau q(\tau) \qquad q(\tau)_{\tau \to \infty} \sim Ai(\tau)$$

- Ramirez, Rider, Virag
- Bloemendal, Virag
- Open problems: Where is Painlevé 2 coming from? Arbitrary β? Generalizations?

$$H = -\frac{d^2}{dt^2} + V(t) + b'(t)$$

Outline

- Introduction: Schrödinger eigenvalue problem
- The white noise Hamiltonian and the Riccati diffusion
- Physical picture $\rightarrow$ tail
- The supersymmetric Hamiltonian
- The Airy Hamiltonian, Kramers regime and semi-classical analysis (A.C, C. Hagendorf)

Schrödinger eigenvalue problem on $L^2(0, L)$

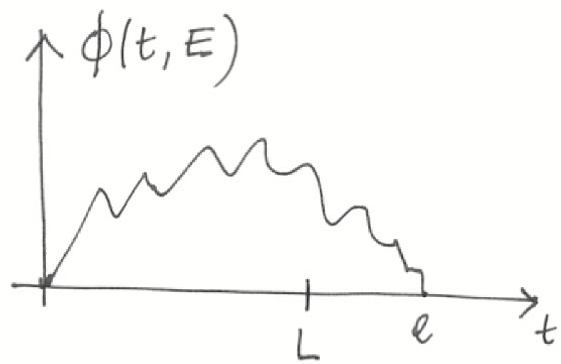
$$H\phi_0(t) = E_0\phi_0(t)$$

$$\phi_0(0) = \phi_0(L) = 0$$

Sturm-Liouville problem → Initial value problem

$$H\phi(t, E) = E\phi(t, E)$$

$$\begin{cases} \phi(0, E) = 0 \\ \phi'(0, E) = 1 \end{cases}$$



- First zero $\phi(l, E) = 0$
- Oscillation theorem $l > L \iff E_0(L) > E$
- H random, distribution of $E_0(L)$ → first passage problem

The white noise Hamiltonian and the Riccati diffusion

$$H = -\frac{d^2}{dt^2} + V(t)$$

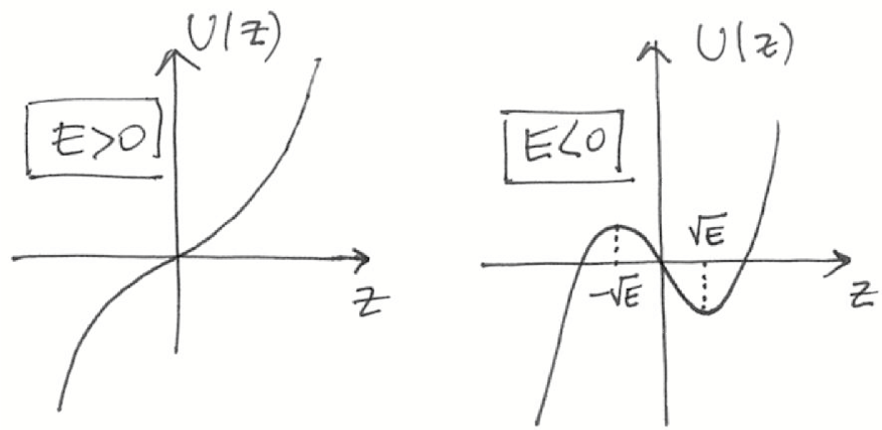
$$\mathbb{E}(V(t)V(t')) = \sigma\delta(t - t')$$

Riccati diffusion

$$z(t) := \frac{\phi'(t, E)}{\phi(t, E)}$$

$$\frac{dz}{dt} = -E - z^2 + V(t) = -\frac{d}{dz}U(z) + V(t)$$

Potential $U(z) = \frac{z^3}{3} + Ez$



The hitting time distribution

$\phi(l, E) = 0 \Rightarrow z(l) = -\infty$, thus l is the first hitting time for a process starting from $z(0) = z$

$$l(z) = \inf(t > 0, z(t) = -\infty | z(0) = z)$$

$$\mathbb{P}[E_0(L) > E] = \mathbb{P}[l(z) > L]$$

- Characteristic function

$$h(z, \alpha) := \mathbb{E}(e^{-\alpha l(z)})$$

$$\left(\frac{\sigma}{2}\frac{d^2}{dz^2} - (E + z^2)\frac{d}{dz}\right) h(z, \alpha) = \alpha h(z, \alpha), \quad h(-\infty, \alpha) = 1$$

- Invariant measure

$$\left(\frac{\sigma}{2}\frac{d}{dz} + (E + z^2)\right) T(z) = N(E) \Rightarrow \mathbb{E}(l(\infty)) = \frac{1}{N(E)}$$

- McKean, Molcanov

Physical picture $E\sigma^{-2/3} \to -\infty$

- Barrier $\Delta U = \frac{4}{3}E^{3/2}$
- Mean exit time

$$\mathbb{E}(l) = \tau \sim \frac{2\pi}{(U''(a)U''(b))^{1/2}} e^{\frac{\Delta U}{D}} = \frac{\pi}{\sqrt{E}} \exp \frac{8E^{3/2}}{3\sigma} = \frac{1}{N(E)}$$

- Exit time distribution $P(l) = \frac{1}{\tau} e^{-\frac{l}{\tau}}$ (weak noise)
- Ground state energy distribution

$$\mathbb{P}[E_0(L) > E] = \mathbb{P}[l > L] = \int_L^\infty dl\, P(l) = \exp\left(-\frac{L\sqrt{E}}{\pi} e^{-\frac{8E^{3/2}}{3\sigma}} \right)$$

- Limiting distribution

$$\mathbb{P}\left[\frac{E_0 + (\frac{3}{8}\log\frac{L}{\pi})^{2/3}}{(24\log L)^{-1/3}} < x \right] = 1 - e^{-e^x}$$

Generalization (C.Texier)

- Exit time distribution $P(l) = N(E)e^{-lN(E)}$
- Sturm
  $\mathbb{P}[E_{n-1} < E < E_n] = \mathbb{P}[l_1 + l_2 + ... + l_n < L < l_1 + l_2 + ... + l_{n+1}]$
- l_i are i.i.d

$$\mathbb{P}[E_{n-1} < E < E_n] = \frac{(LN(E))^n}{n!} e^{-LN(E)}$$

$$\mathbb{P}\left[\frac{E_{n-1} - f_{n-1}(L)}{\sigma_{n-1}(L)} \in dx\right] = \frac{n^{n-1/2}}{(n-1)!} \exp\left(x\sqrt{n} - ne^{x/\sqrt{n}}\right) dx$$

- Lesson Trapping of the Riccati variable $\Rightarrow$ Gumbel distribution
  Eigenvalues are uncorrelated (Cf Molcanov)

The supersymmetric Hamiltonian

Different formulations

- Random scalar Hamiltonian

$$H = (\partial_x + \Phi)(-\partial_x + \Phi) = Q^+ Q$$

$$\mathbb{E}(\Phi(x)\Phi(y)) = g\delta(x-y)$$

  Interesting feature: existence of a delocalized state for $E = 0$
  $N(E) \sim \frac{2g}{\ln^2(g^2/E)} \quad \gamma(E) \sim \frac{2g}{\ln(g^2/E)}$

- Dirac Hamiltonian $\begin{cases} Q\phi = k\chi \\ Q^+\chi = k\phi \end{cases}$

- Classical diffusion in a random force $\frac{dx(t)}{dt} = \Phi(x(t)) + \eta(t)$

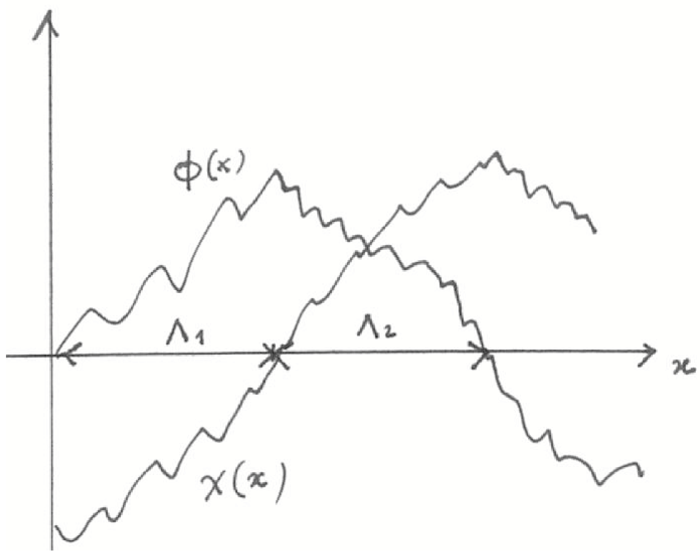
- Wider class of models based on Levy processes (A.C, C. Texier, Y. Tourigny)

Dirac formulation (C. Texier)

$$\begin{cases} (-\frac{d}{dx} + \Phi)\phi = k\chi \\ (\frac{d}{dx} + \Phi)\chi = k\phi \end{cases}$$

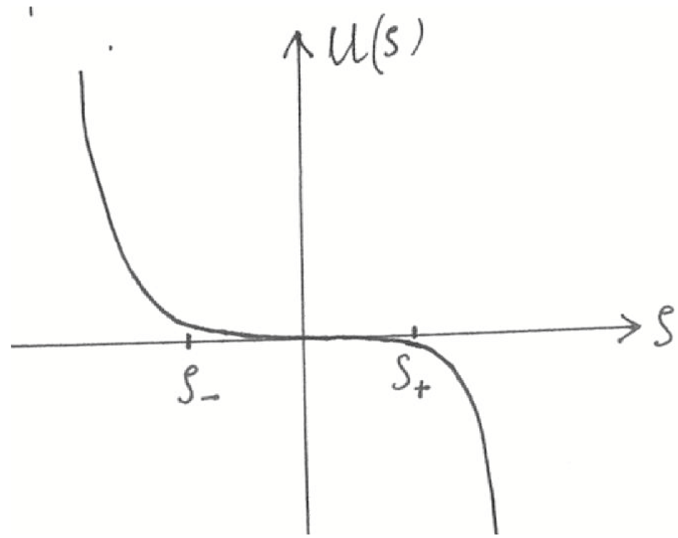
- Riccati analysis $\begin{cases} \phi = e^{\xi}\sin\theta \\ \chi = -e^{\xi}\cos\theta \end{cases}$

- Alternation of nodes : $l = \Lambda_1 + \Lambda_2$



- $\zeta = \frac{1}{2} \ln \tan(\theta)$

$$\frac{d\zeta}{dx} = k \cosh(2\zeta) + \Phi = -\frac{\partial U}{\partial \zeta} + \Phi$$



- Free diffusion on $[\zeta_-, \zeta_+] = [-\frac{1}{2} \ln(\frac{g}{k}), \frac{1}{2} \ln(\frac{g}{k})]$
- $\mathbb{E}(l) = \frac{2}{g}(\zeta_+ - \zeta_-)^2 = \frac{2}{g} \ln^2(\frac{g}{k})$

- Λ_1 = First hitting time of ζ_+
- Characteristic function $h(\zeta,\alpha) = \mathbb{E}(e^{-\alpha\Lambda_1})$ satisfies

$$\left(\frac{g}{2}\frac{d^2}{d\zeta^2} + k\cosh 2\zeta\right) h(\zeta,\alpha) = \alpha h(\zeta,\alpha)$$

- Low energy limit $h(\zeta_-,\alpha) = \frac{1}{\cosh\sqrt{\frac{\alpha}{N(E}}}$

$$\mathbb{E}\left(e^{-\alpha(\Lambda_1+\Lambda_2)N(E)}\right) = \frac{1}{\cosh^2\sqrt{\alpha}}$$

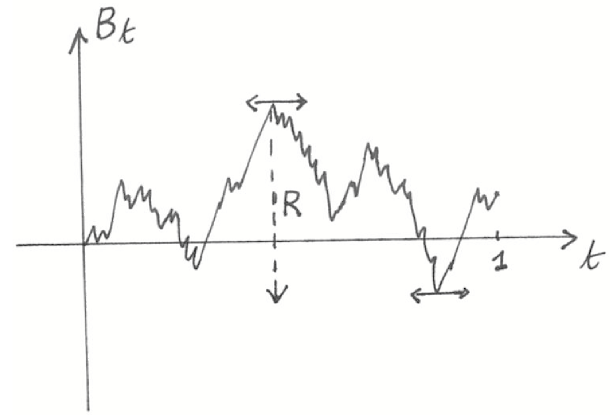
- Density (Cf. P. Le Doussal, C. Monthus)

$$f(x = lN(E)) = \frac{4}{\sqrt{\pi}x^{3/2}}\sum_{n=1}^{\infty}(-1)^{m+1}m^2 e^{-\frac{m^2}{x}}$$

- In which sense is it an extreme value distribution?

The ground state energy and the range of a Brownian motion

- A reformulation: the density of $y = \sqrt{\frac{2}{lN(E)}}$ is the density of the range



$$\mathbb{R} = \sup_{0\leqslant t,t'\leqslant 1} (B_t - B_{t'})$$

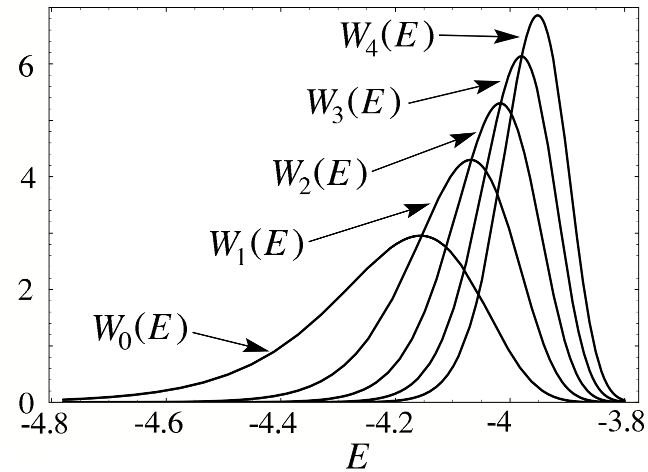
$$g_{\mathbb{R}}(y) = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (-1)^k k^2 e^{-\frac{k^2 x^2}{2}}$$

- An identity in law

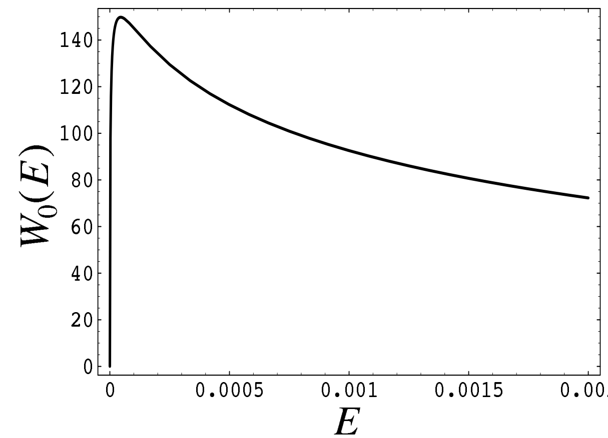
$$\sup_{0\leqslant t,t'\leqslant 1} (B_t - B_{t'}) = -\frac{1}{\sqrt{gL}} \ln \frac{E_0(L)}{g^2}$$

- Elements of a proof (A.C., G. Oshanin, C. Monthus)

White noise hamiltonian

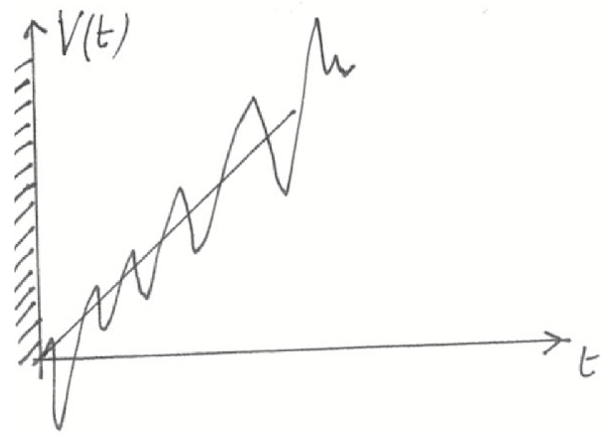


Supersymmetric hamiltonian



The Airy Hamiltonian

$$H = -\frac{d^2}{dt^2} + t + \eta(t)$$
$$\mathbb{E}\left[\eta(t)\eta(t')\right] = \sigma\delta(t - t')$$



Boundary condition $\psi'(0) = z_0\psi(0)$

Riccati dynamics $\frac{dz}{dt} = -z^2 + t - E + \eta(t)$

Initial condition $z(0) = z_0$

Shift $t - E \to t$

$$\frac{dz}{dt} = -z^2 + t + \eta(t)$$

Initial condition $z(-E) = z_0$

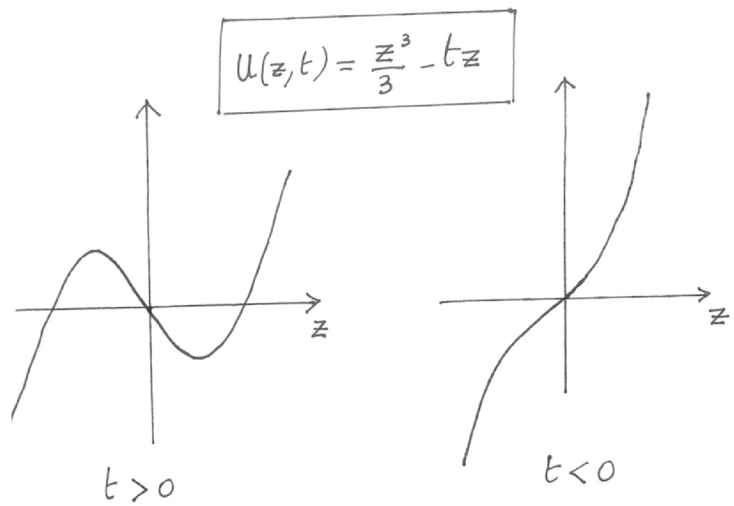
Distribution of the ground state energy?

$$F(z_0, E) := [\mathbb{P}(E < E(z_0)] = [\mathbb{P}(z(t) \text{ does not explode } \forall t > 0]$$

Physical picture Dynamics of the Riccati variable with

⋆ a stable point $z_0 = \sqrt{t}$

⋆ an unstable point $z_1 = -\sqrt{t}$.



- Particle is dropped at $t_0 = -E$.
- For $E > 0, t_0 < 0$ the system has not yet reached equilibrium.
- For $E < 0, t_0 > 0$ the system is already equilibrated: Kramers like regime.

Activation dynamics in a time dependent potential $U(z,t) = U(z) - tz$

A guide: the case $\beta = 2$! $\longrightarrow$ TW distribution

$$F(\infty, E) = \exp - \int_{-E}^{\infty} (\tau + E) q^2(\tau) d\tau$$

The Kramers regime $E \ll 0$: adiabatic approximation

Escape rate

$$k(t) = \frac{\omega_a \omega_b}{2\pi} \exp - \frac{\Delta U}{D} = \frac{t^{1/2}}{\pi} \exp - \frac{2\beta t^{3/2}}{3}$$

Rate equation

$$\frac{\partial P}{\partial t}(t|t_0) = -k(t) P(t|t_0)$$

Survival probability for infinite time

$$F(E) = P(\infty, t_0) = \exp\left(- \int_{t_0}^{\infty} k(t) dt \right) = \exp\left(- \frac{1}{\pi\beta} e^{-\frac{2\beta(-E)^{3/2}}{3}} \right)$$

Reproduces the right exponential tail (Cf S. Majumdar, M.Vergassola, 2008) but not the prefactor (Dumas-Virag, 2011)

$$F(E) \sim 1 - \frac{1}{\pi\beta} \exp - \frac{2\beta(-E)^{3/2}}{3}, \quad F(E) \sim 1 - \frac{c_\beta}{(-E)^{3\beta/4}} e^{-\frac{2\beta(-E)^{3/2}}{3}}$$

Semiclassical analysis

Fokker-Planck equation satisfied by $P(zt|z_0t_0)$

$$\frac{\partial P}{\partial t} = \frac{\sigma}{2}\frac{\partial^2 P}{\partial z^2} + \frac{\partial}{\partial z}[(z^2 - E)P]$$

Small noise limit $\sigma = \frac{4}{\beta} \to 0$

Set $P = \exp -\beta S$

$$\frac{\partial S}{\partial t} = \frac{2}{\beta}\left(\frac{\partial^2 S}{\partial z^2} - z\right) - 2\left(\frac{\partial S}{\partial z}\right)^2 + (z^2 - t)\frac{\partial S}{\partial z}$$

WKB expansion $S = \sum_{n=0}^{\infty} \frac{S_n}{\beta^n}$

- $n = 0$ $\quad \frac{\partial S_0}{\partial t} + 2\left(\frac{\partial S_0}{\partial z}\right)^2 + (t - z^2)\frac{\partial S_0}{\partial z} = 0$
- Hamilton Jacobi equation $\frac{\partial S}{\partial t} + H(\frac{\partial S}{\partial t}, z, t) = 0$

Hamiltonian

$$H(p,q,t) = 2p^2 + p(t - q^2)$$

Action along the classical path $z = q(t)$

$$S_0(zt|z_0t_0) = \frac{1}{8}\int_{t_0}^{t} (\dot{q} + q^2 - \tau)^2 d\tau$$

Equation of motion

$$\begin{cases} \dot{p} = 2pq \\ \dot{q} = 4p + t - q^2 \end{cases}$$

Non explosion probability

$$F(z_0, E) = \lim_{t\to\infty} \int_{-\sqrt{t}}^{\infty} \frac{dz}{\psi(t)} \exp -\beta S_0(zt|z_0t_0)$$

Saddle point approximation $\frac{\partial S_0}{\partial z}|_{z=z*} = 0 \Rightarrow p(t) \to 0$

One gets $q(t) \sim \frac{Ai'(t)}{Ai(t)} \sim -\sqrt{t}$ and $p(t) \sim Ai(t)^2 \sim \exp -\frac{4t^{3/2}}{3}$

Analysis of the classical paths

- What are the classical paths with appropriate boundary condition for $t \to \infty$?
- Set $q = \frac{v^2}{2}$

$$\begin{cases} \dot{v} = vq \\ \dot{q} = 2v^2 + t - q^2 \end{cases}$$

$\Rightarrow$The Painlevé **II** equation !

$$\boxed{\ddot{v} = 2v^3 + tv}$$

- Action

$$F(z_0, E) \sim e^{-\beta S_0} = \exp -\frac{\beta}{8}\int_{-E}^{\infty} d\tau\,(\dot{q}+q^2-\tau)^2 = \exp -\frac{\beta}{2}\int_{-E}^{\infty} d\tau\, v^4(\tau)$$

- Hastings McLeod solution $v(t)_{t\to\infty} \sim Ai(t)$

Conclusion :

- Within the semi-classical approach $\beta \gg 1$, the Hastings McLeod solution seems to describes the regime $E \to \infty$ with Neumann boundary condition at the origin $z_0 = 0$ leading to the left tail behaviour

$$F(0, E) \sim \exp -\frac{\beta E^3}{24}$$

Compare with

- Dean-Majumdar (2008) (Coulomb gas approach)
- Borot, Eynard, Majumdar, Nadal : fixed β and $E \to \infty$

$$F(\infty, E) \sim \frac{\exp -[\beta \frac{E^3}{24} - \frac{\sqrt{2}}{3}(\frac{\beta}{2} - 1)E^{3/2}]}{E^{\frac{1}{8}(3 - \frac{2}{\beta} - \frac{\beta}{2})}}$$

Conclusion

- Relation between ground state energy distribution and exit time distribution from a potential barrier.
- Role of the potential (time dependance)$\rightarrow$ Different extreme value distributions. Qualitative description of the tails.
- Role of the noise?
- Extension to a general Levy process where the jump distribution is encoded in the Levy measure.
- Deterministic diffusion $\rightarrow$ inverse local time $\rightarrow$ subordinator $\rightarrow$ random potential (A.C., C. Texier, Y. Tourigny)

- $x_1, x_2, ..., x_n$ set of i.i.d random variables drawn from $p(x)$
  $\rightarrow P(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} p(x_i)$
- Maximum $x_{\rm max} = \max(x_1, x_2, ..., x_n)$
- $Q_n(x) = \mathbb{P}[x_{\rm max} \leqslant x] = [\int_{-\infty}^{x} p(t)dt]^n$
- Scaling limit : n large, x large: $Q_n(x) \rightarrow F[\frac{x-a_n}{b_n}]$
  a_n, b_n non universal
- 3 limiting distributions $F(z)$ depending only on the tails of $p(x)$

$$\begin{cases} \text{Gumbel} \\ \text{Frechet} \\ \text{Weibull} \end{cases}$$