

Order statistics in $1/f^\alpha$ signals

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with N. Moloney and K. Ozogány

Order statistics:

What is it and why is it interesting?

Astrophysical examples:

Importance of correlations:

$1/f^\alpha$ process as a simple model system.



Simulations:

Emergence of power law spectra

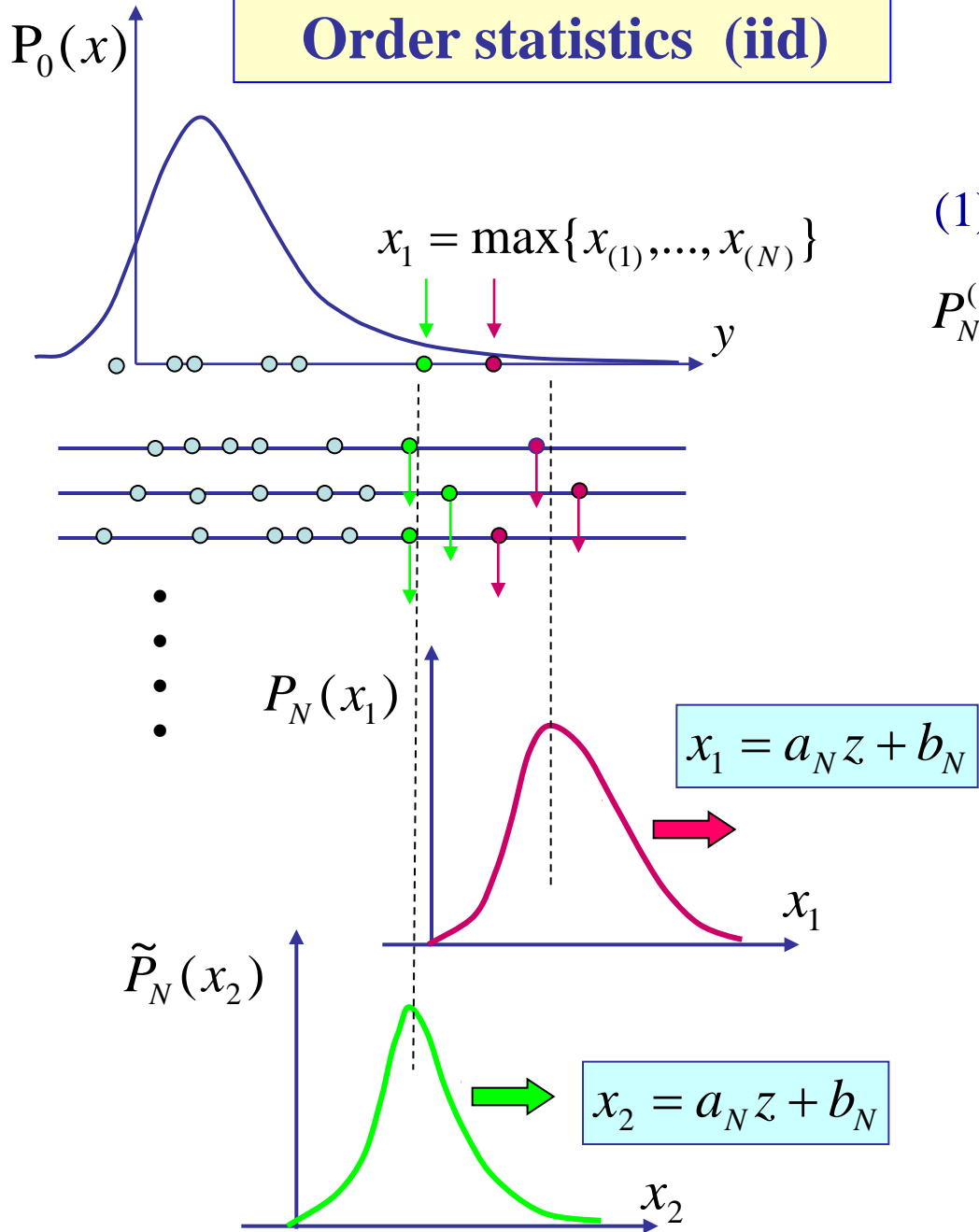
Scaling regimes: $0 \leq \alpha < 1$; $1 < \alpha \leq 5$; $5 \leq \alpha < \infty$

Phenomenology:

Scaling arguments and derivation of the exponents

Memoriter: Comparison with quantum spectra

Order statistics (iid)



What are we interested in?

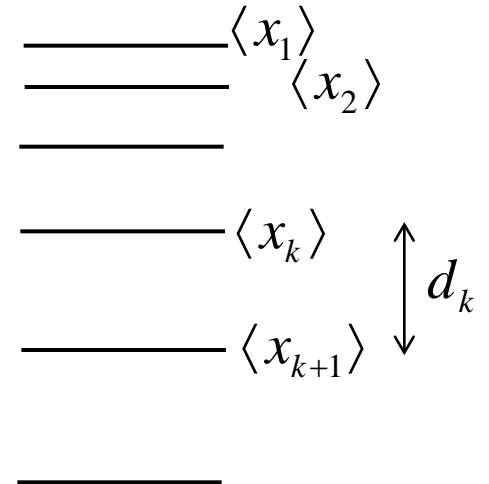
(1) Limit distributions

$$P_N^{(k)}(x_k = a_N z + b_N) \xrightarrow{N \rightarrow \infty} \Phi_k(z)$$

(2) Gap (scaled or unscaled)

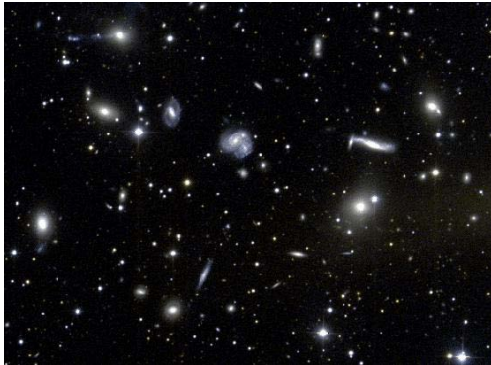
$$d_1 = \langle x_1 - x_2 \rangle$$

(3) Spectrum



Motivation: Astrophysical problems

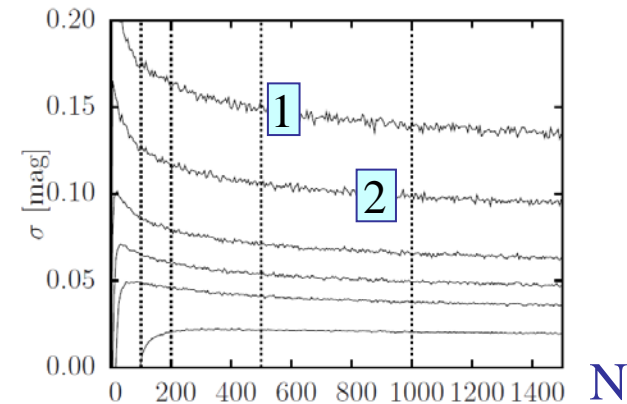
Standard candles: Using the brightest or the 2nd brightest, or the 3rd ...?



Hercules Cluster

$N \sim 100$

Dobos-Csabai, 2011
St. dev. of magnitude

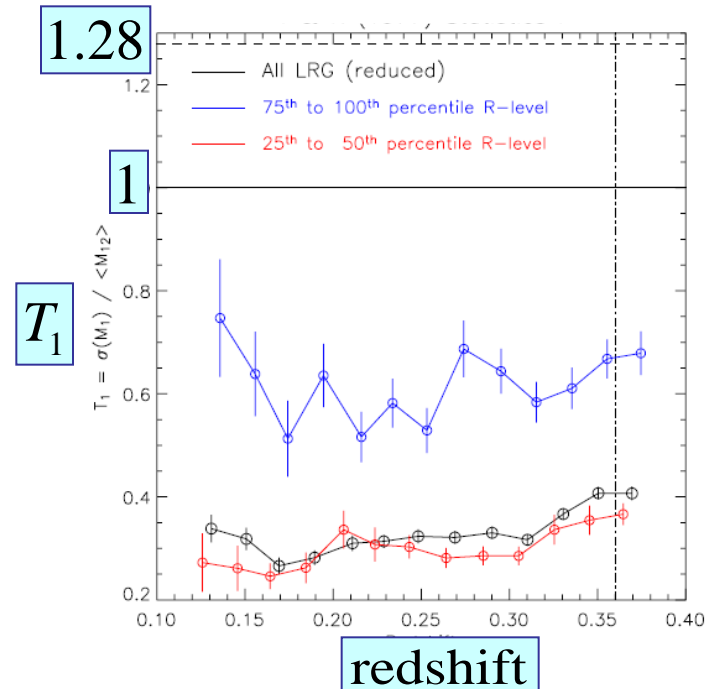


Testing for iid:

Tremain-Richstone, 1977

$$T_1 = \frac{\sqrt{\langle x_1^2 \rangle - \langle x_1 \rangle^2}}{\langle x_1 - x_2 \rangle} \geq 1$$

iid Gumbel class: $T_1 = 1.28$



Loh
-Strauss
2006

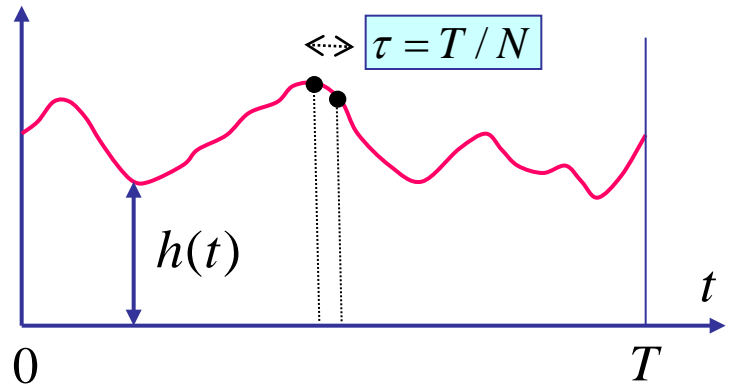
Gaussian $1/f^\alpha$ signals

N independent, nonidentically distributed Fourier modes

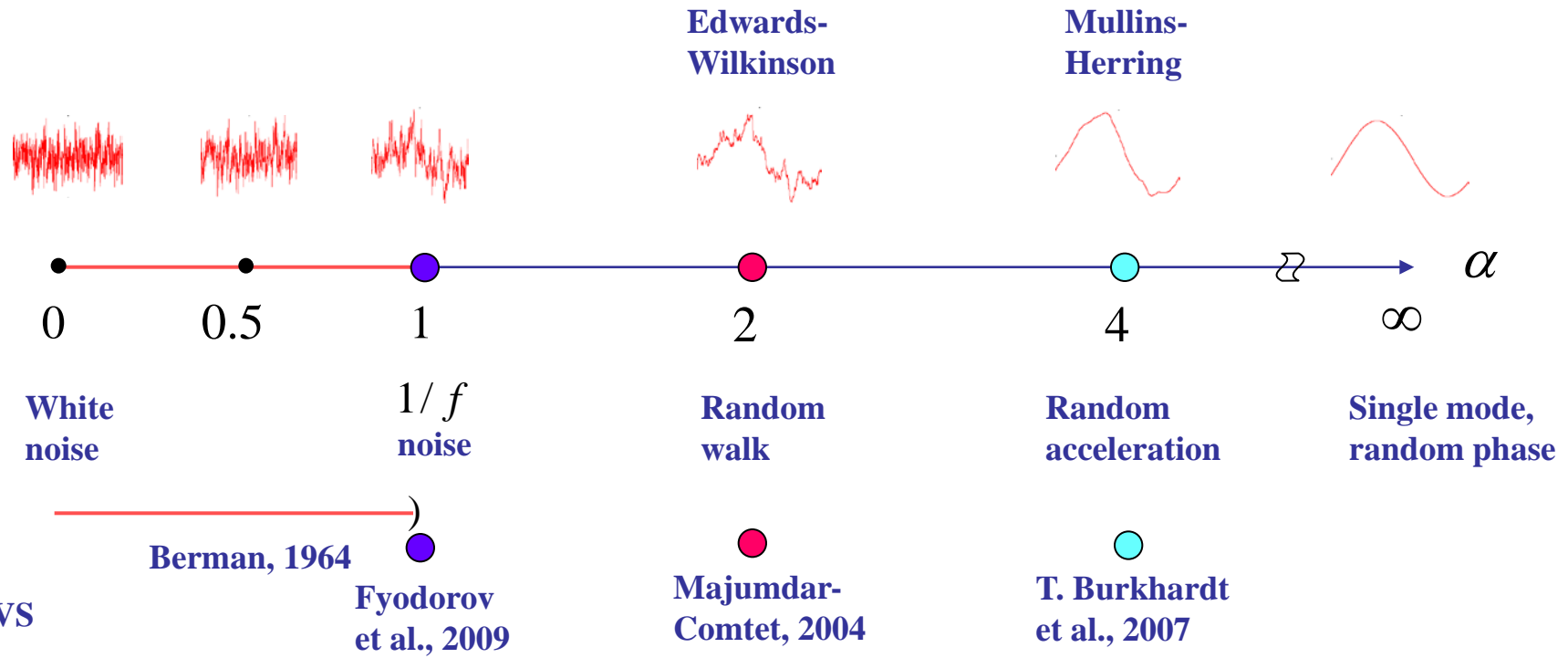
$$P(\{h_q\}) \sim e^{-\sigma_T \sum_q^{(N)} |q|^\alpha |h_q|^2}$$

with singular fluctuations

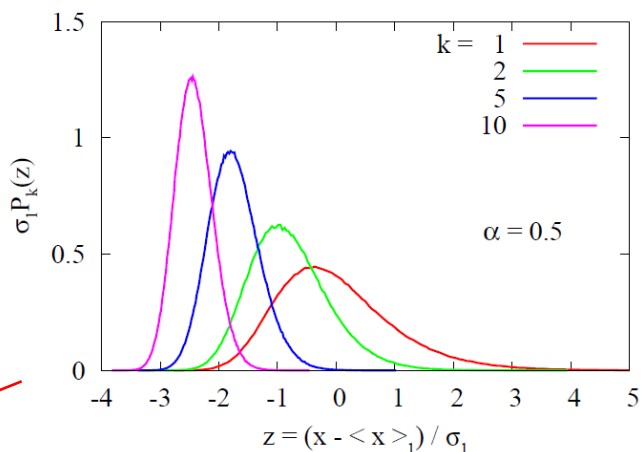
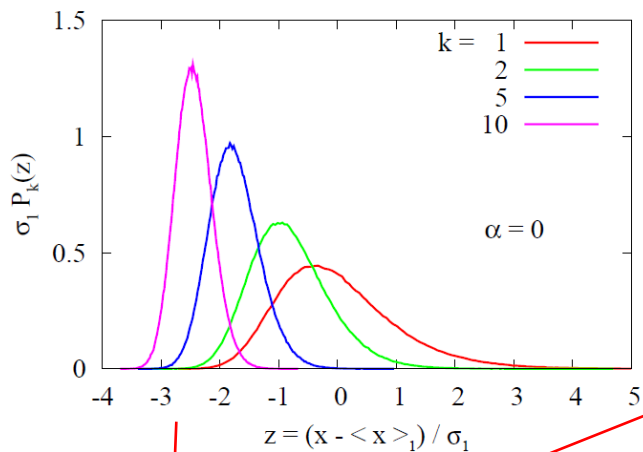
$$\langle |h_q|^2 \rangle \sim q^{-\alpha}$$



$$\langle w^2 \rangle = \overline{(h - \bar{h})^2} = T^{-1} \sum_q \langle |h_q|^2 \rangle \sim T^{\alpha-1}$$



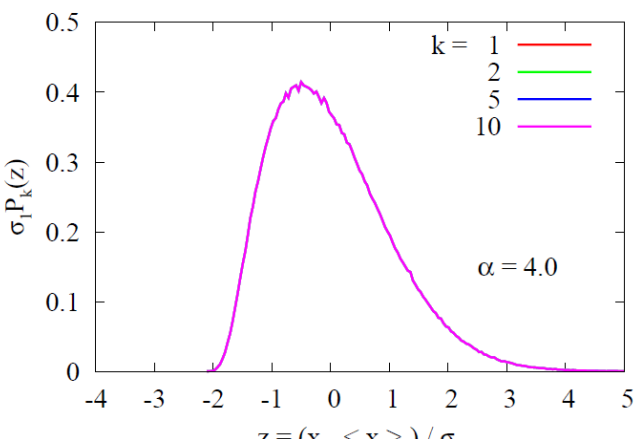
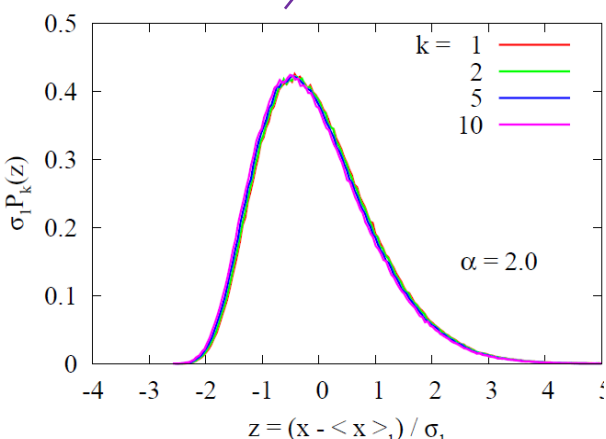
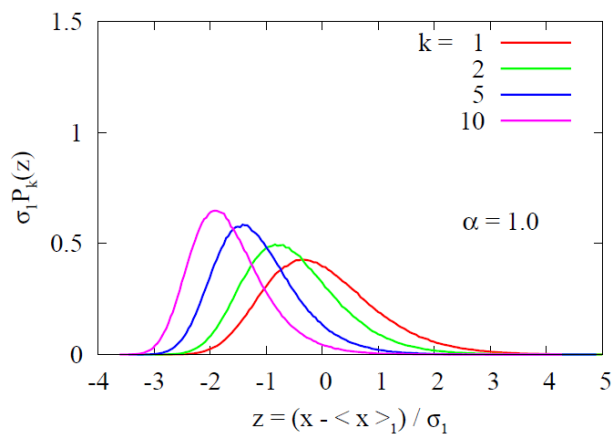
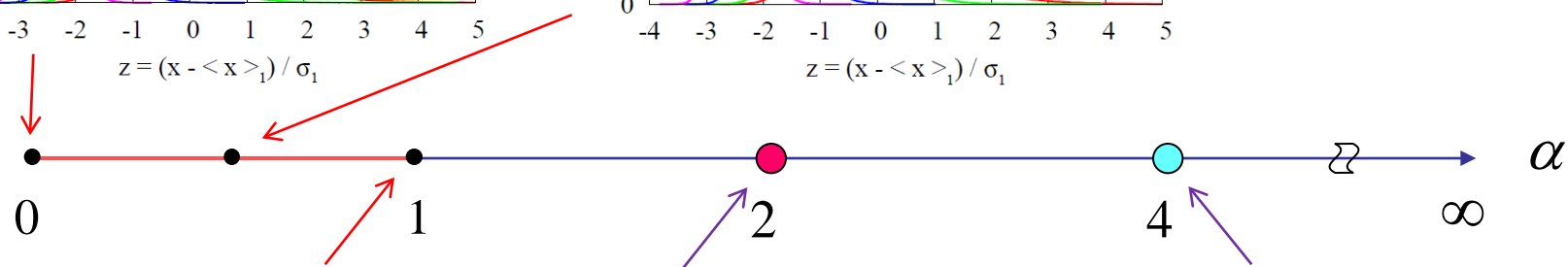
Distribution of 1st, 2nd, ..., 10th



$$\alpha < 1$$

$$\langle x_k \rangle \sim \sqrt{\ln N}$$

$$\langle \sigma_k \rangle \sim 1/\sqrt{\ln N}$$



$$\langle x_k \rangle \sim \langle \sigma_k \rangle \sim N^{(\alpha-1)/2}$$

$$\alpha > 1$$

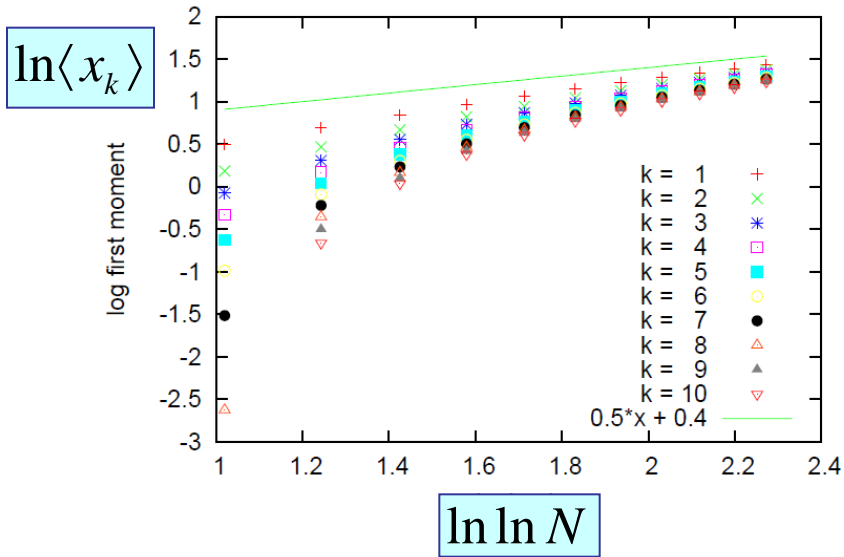
Györgyi et al. 2007

Averages of 1st, 2nd, ..., 10th

System sizes: $N = 16, 32, \dots, 16384$

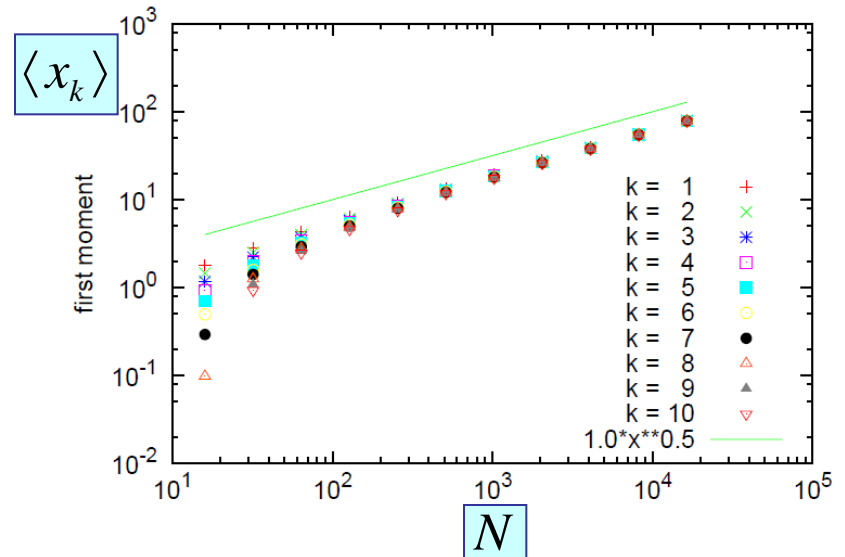
Number of runs: 1 000 000

$\alpha = 0.5$



$$\langle x_k \rangle \sim \sqrt{\ln N}$$

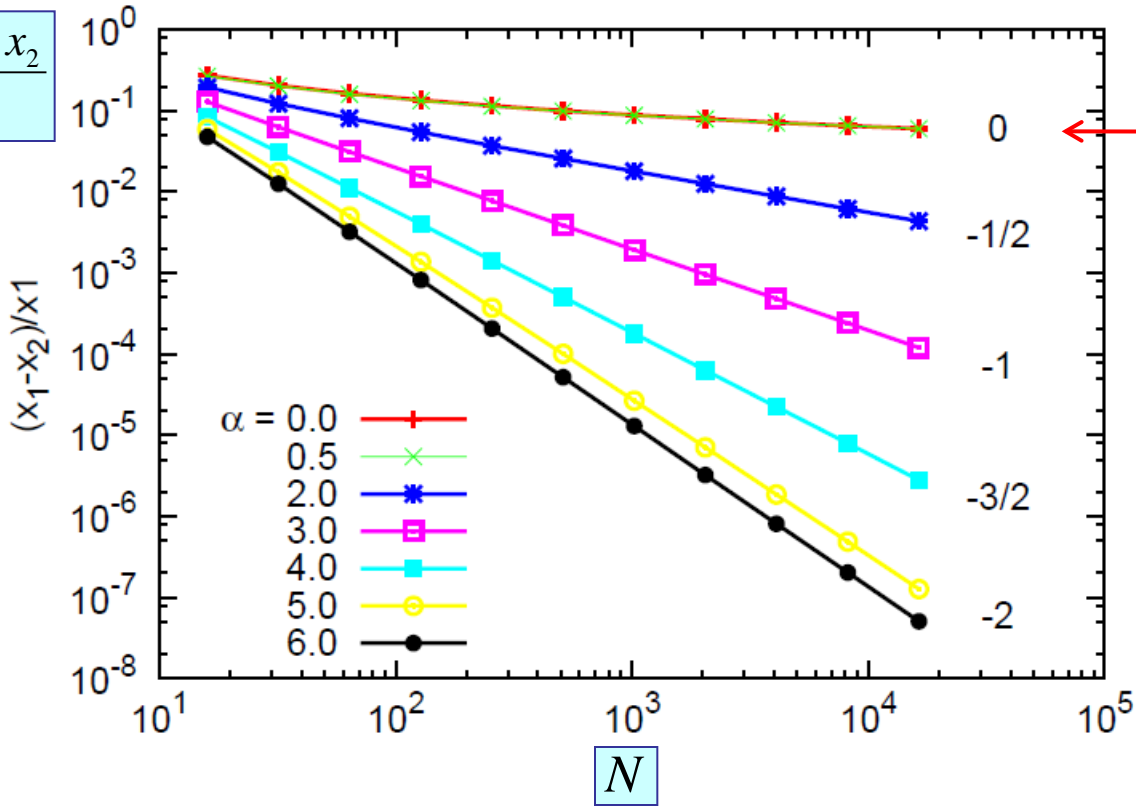
$\alpha = 2$



$$\langle x_k \rangle \sim N^{1/2}$$

Gap (scaled): I

$$\frac{x_1 - x_2}{x_1}$$

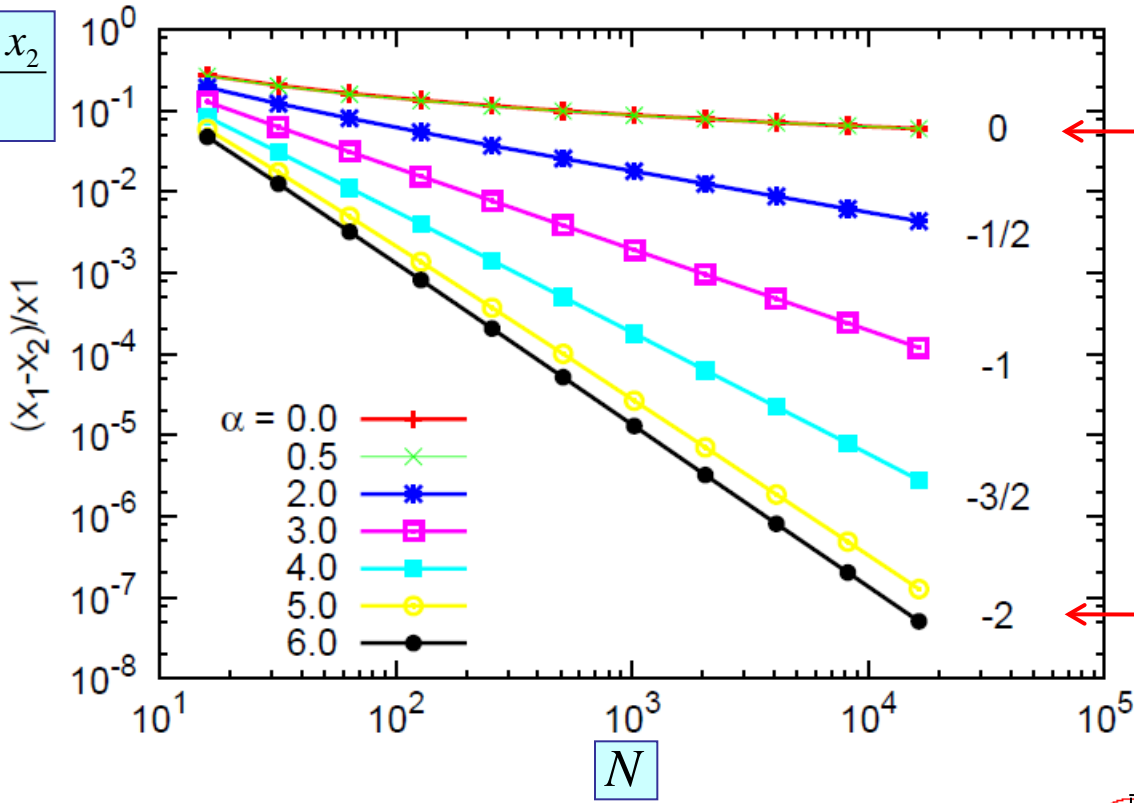


$$\frac{x_1 - x_2}{x_1} \approx \frac{1}{\ln N}$$

$$\alpha = 0 \quad \rightarrow \quad x_1 \approx \frac{1}{\sqrt{\ln N}} z + \sqrt{\ln N} \quad \rightarrow \quad x_1 \approx \sqrt{\ln N} \quad \rightarrow \quad x_1 - x_2 \approx \frac{1}{\sqrt{\ln N}}$$

Gap (scaled): II

$$\frac{x_1 - x_2}{x_1}$$



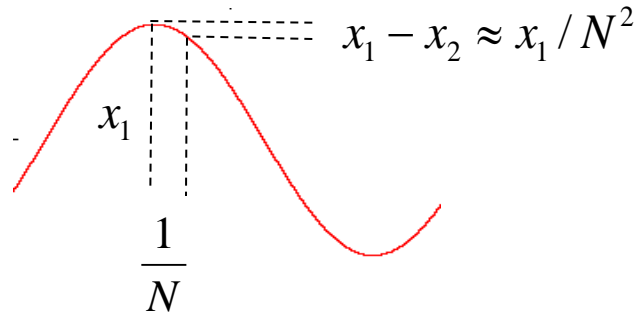
$$\frac{x_1 - x_2}{x_1} \approx \frac{1}{\ln N}$$

$$\frac{x_1 - x_2}{x_1} \approx \frac{1}{N^2}$$

$$\frac{N^{(\alpha-5)/2}}{N^{(\alpha-1)/2}}$$

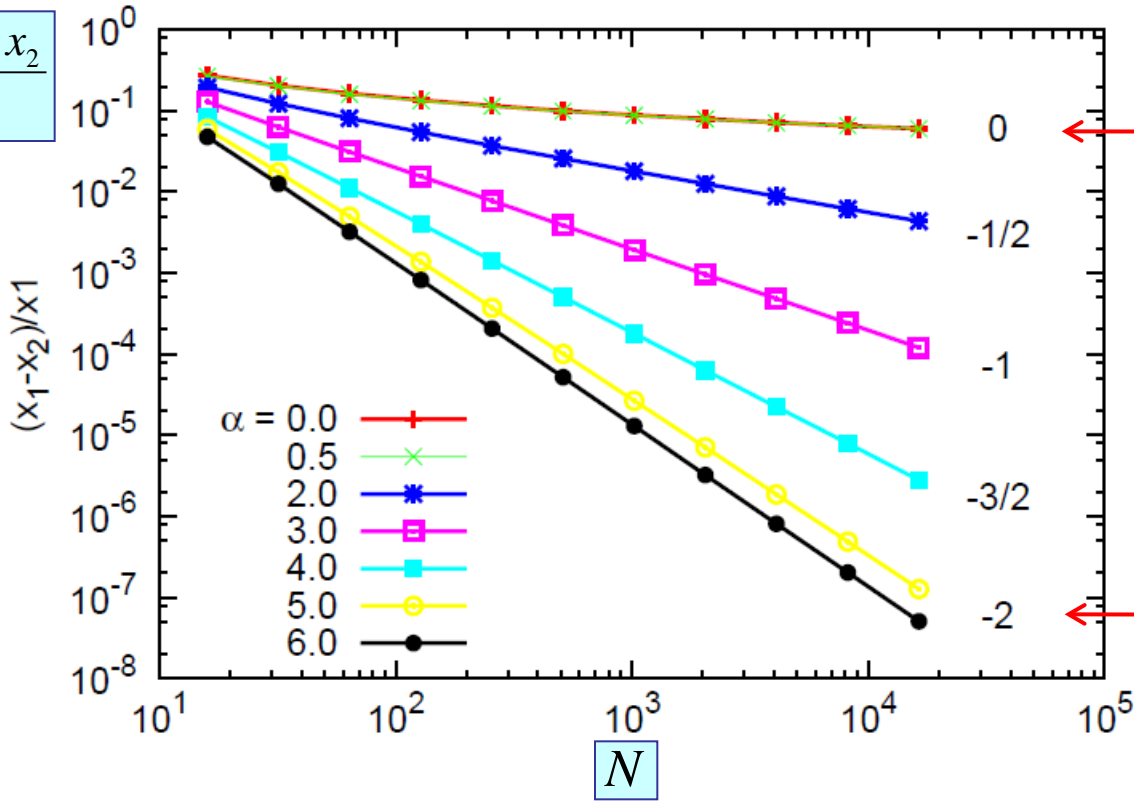
$\alpha \geq 5 \rightarrow$ 2nd derivative exists

$\alpha = \infty$



Gap (scaled): III

$$\frac{x_1 - x_2}{x_1}$$



$$\frac{x_1 - x_2}{x_1} \approx \frac{1}{\ln N}$$

$0 \leq \alpha < 1$

$$\frac{x_1 - x_2}{x_1} \approx \frac{\text{const}}{N^{(\alpha-1)/2}}$$

$1 < \alpha < 5$

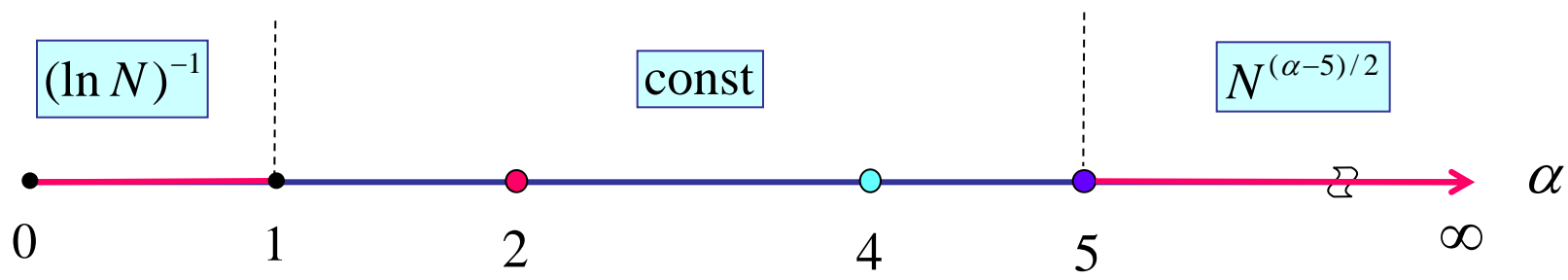
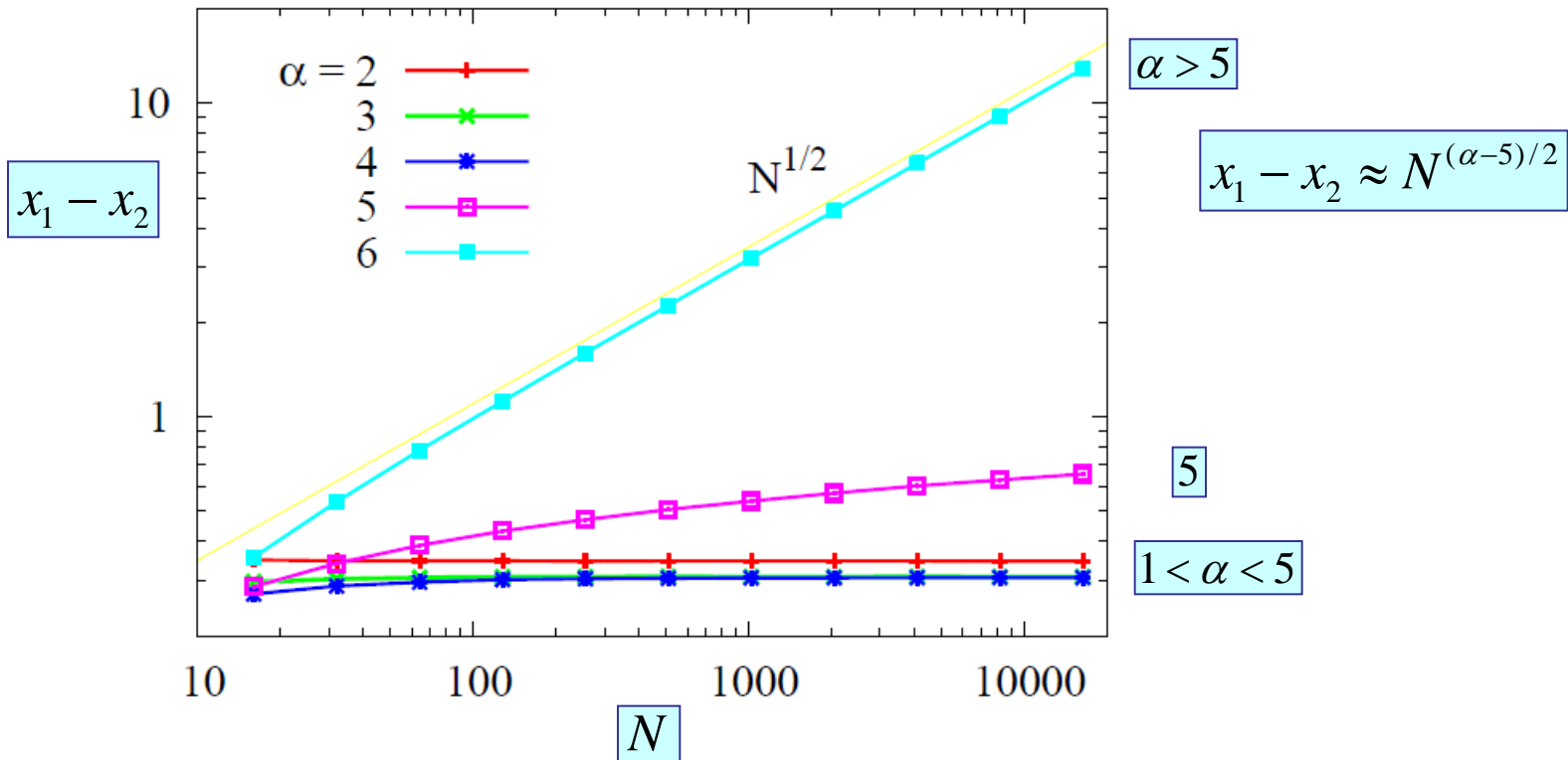
$$\frac{x_1 - x_2}{x_1} \approx \frac{1}{N^2}$$

$5 < \alpha < \infty$

$$\frac{N^{(\alpha-5)/2}}{N^{(\alpha-1)/2}}$$

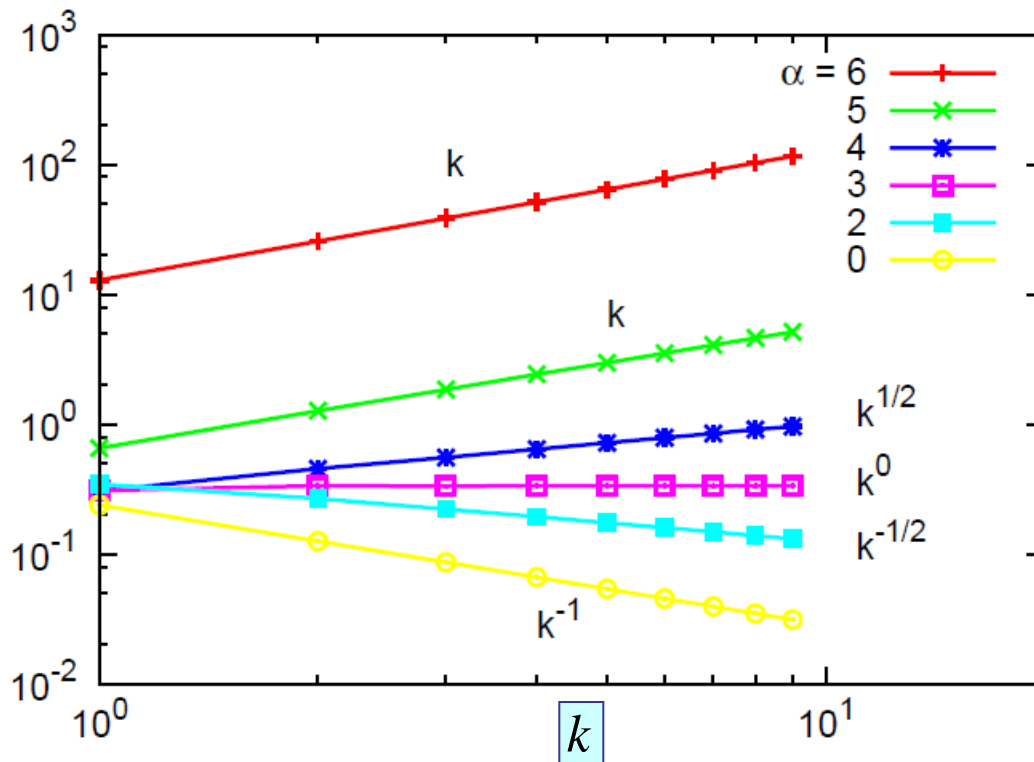
$1 < \alpha \leq 5 \rightarrow x_1 \approx N^{(\alpha-1)/2}$ and $\frac{x_1 - x_2}{x_1} \approx \frac{1}{N^{(\alpha-1)/2}} \rightarrow x_1 - x_2 \approx \text{const}$

Gap (unscaled)



Gap between the k-th and k+1st largest

$$d_k = x_k - x_{k+1}$$



$$0 \leq \alpha < 1$$

$$d_k \sim k^{-1}$$

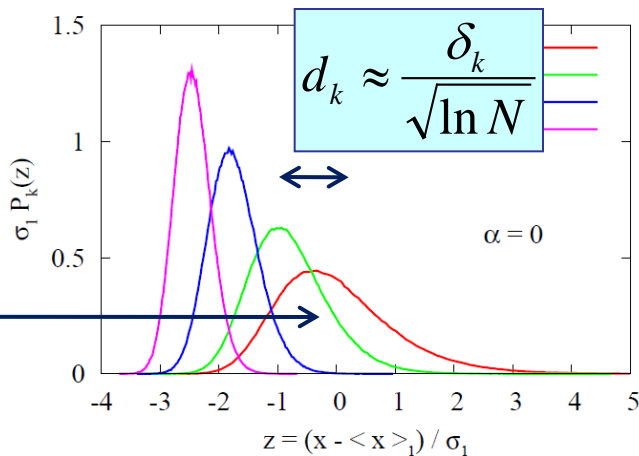
$$1 < \alpha < 5$$

$$d_k \sim k^{(\alpha-3)/2}$$

$$5 < \alpha$$

$$d_k \sim k$$

Phenomenology for $d_k = x_k - x_{k+1}$



$$b_N \sim \sqrt{\ln N}$$

$$0 \leq \alpha < 1$$

$$\sqrt{\ln N} \approx b_N = \sum_{k=1}^N d_k = \sum_{k=1}^N \frac{\delta_k}{\sqrt{\ln N}}$$

assumption

$$\sum_{k=1}^N \delta_k \approx \ln N \rightarrow \delta_k \sim k^{-1}$$

$$1 < \alpha < 5$$

$$b_N \sim N^{(\alpha-1)/2}$$

$$d_k \approx \text{const} \cdot \delta_k$$

$$\sum_{k=1}^N \delta_k \approx N^{(\alpha-1)/2} \rightarrow \delta_k \sim k^{(\alpha-3)/2}$$

$$5 < \alpha$$

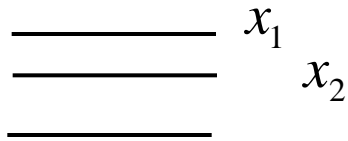
$$b_N \sim N^{(\alpha-1)/2}$$

$$d_k \approx N^{(\alpha-5)/2} \delta_k$$

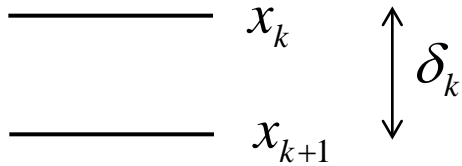
$$\sum_{k=1}^N \delta_k \approx N^2 \rightarrow \delta_k \sim k$$

$$N^{(\alpha-5)/2} \sum_{k=1}^N \delta_k \approx N^{(\alpha-1)/2}$$

Spectrum: $\varepsilon_k = x_1 - x_k$ scaled appropriately .



$$\varepsilon_n = \sum_{k=1}^n \tilde{d}_k = \sum_{k=1}^n \delta_k$$



$$0 \leq \alpha < 1$$

$$\sum_{k=1}^N \delta_k \approx \ln N$$



$$\varepsilon_n \sim \ln n$$

$$1 < \alpha < 5$$

$$\sum_{k=1}^N \delta_k \approx N^{(\alpha-1)/2}$$



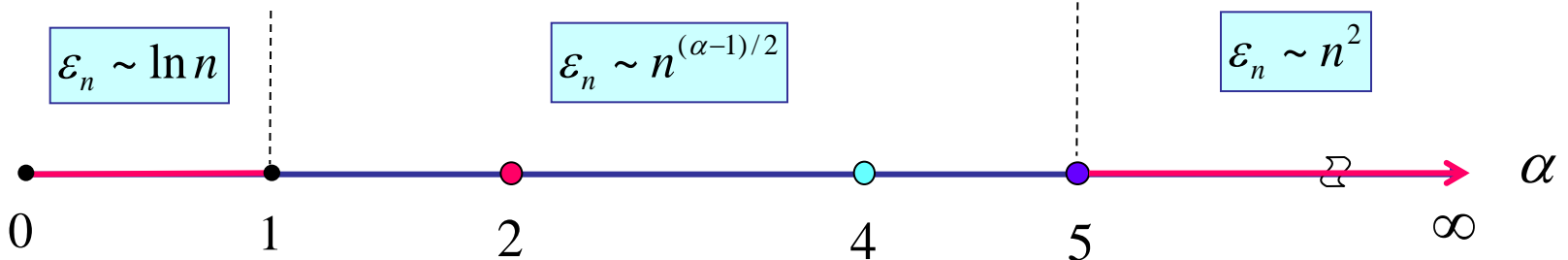
$$\varepsilon_n \sim n^{(\alpha-1)/2}$$

$$5 < \alpha$$

$$\sum_{k=1}^N \delta_k \approx N^2$$

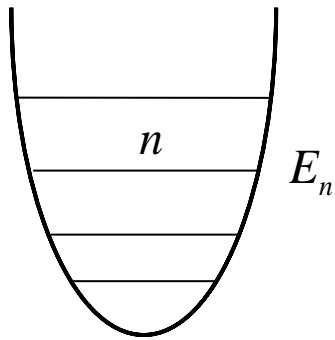


$$\varepsilon_n \sim n^2$$



Comparing with QM spectra

$$U(x) = g |x|^\kappa$$

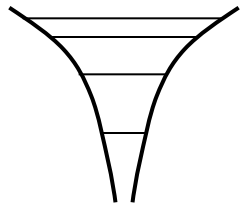


Dimensional analysis: $E_n(m, g, h) \sim h^{\frac{2\kappa}{2+\kappa}}$

Quasi-classical limit: $\oint pdq = nh$

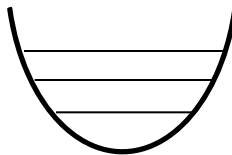
$E_n \sim n^{\frac{2\kappa}{2+\kappa}}$

$$\sim \ln|x|$$

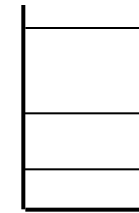


$$0 \leq \alpha < 1 \quad \kappa = 0$$

$$U(x) \sim x^2$$



$$\alpha = 3 \quad \kappa = 2$$

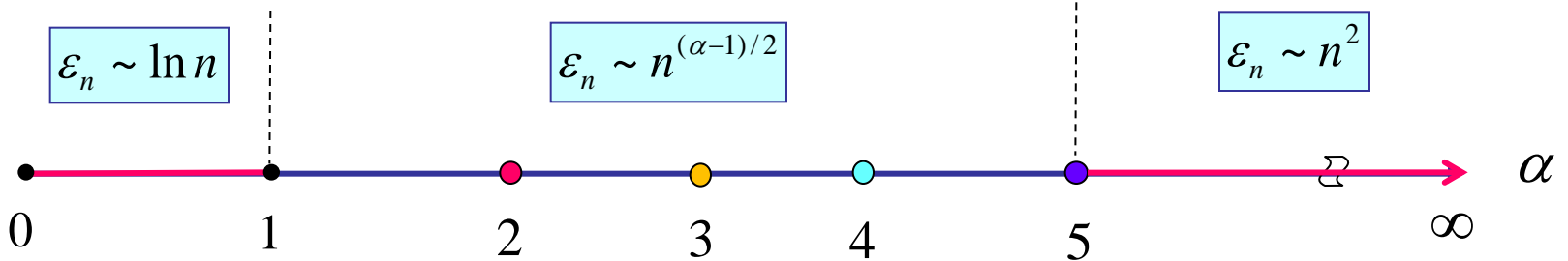


$$5 < \alpha \quad \kappa = \infty$$

$\varepsilon_n \sim \ln n$

$\varepsilon_n \sim n^{(\alpha-1)/2}$

$\varepsilon_n \sim n^2$



Tremain-Richstone ratios

$$T_1 = \frac{\sqrt{\langle x_1^2 \rangle - \langle x_1 \rangle^2}}{\langle x_1 - x_2 \rangle} \geq 1$$

$$T_2 = \frac{\sqrt{\langle (x_1 - x_2)^2 \rangle - \langle x_1 - x_2 \rangle^2}}{\langle x_1 - x_2 \rangle} \geq 0.82$$

