The Distributions of Random Matrix Theory and their Applications

Craig A. Tracy

Department of Mathematics
UC Davis

Extremes and Records
IPhT Saclay
June 14–17, 2011

Outline

- Historical Introduction

Outline

- Historical Introduction
- Random Matrix Models (RMM) with Unitary Symmetry

Outline

- Historical Introduction
- Random Matrix Models (RMM) with Unitary Symmetry
- RMM with Orthogonal Symmetry

Outline

- Historical Introduction
- Random Matrix Models (RMM) with Unitary Symmetry
- RMM with Orthogonal Symmetry
- Universality Theorems

Outline

- Historical Introduction
- Random Matrix Models (RMM) with Unitary Symmetry
- RMM with Orthogonal Symmetry
- Universality Theorems
- Multivariate Statistical Analysis—PCA

 

Figure: E. Ivar Fredholm (1866–1927) and Paul Painlevé (1863–1933).

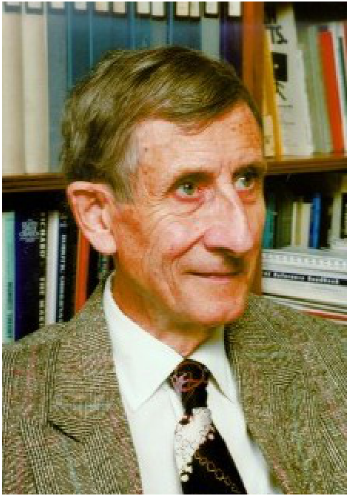

Figure: Eugene Wigner (1902–1995) and Freeman Dyson

We also mention the important early work of

M. L. MEHTA and M. GAUDIN

Gaudin, using Mehta's (then) newly developed polynomial method, was the first to show that the

probability of no eigenvalues in an interval $(0, s)$ in GUE

is expressible as a Fredholm determinant of the *sine kernel*

$$\frac{1}{\pi} \frac{\sin(x-y)}{x-y}$$

evaluated on the interval $(0, s)$

2D Ising Model

First occurence of Toeplitz and Fredholm Dets $\rightsquigarrow$ Painlevé

Wu, McCoy, C.T., & Barouch (1973–77):

$$\lim_{\substack{T \to T_c^{\pm}, R^2 = M^2 + N^2 \to \infty \\ r = R/\xi(T) \text{fixed}}} \mathbb{E}\left(\sigma_{00}\sigma_{MN}\right) = \left\{ \begin{array}{c} \sinh \frac{1}{2}\psi(r) \\ \\ \cosh \frac{1}{2}\psi(r) \end{array} \right\} \times$$

$$\exp\left(-\frac{1}{4}\int_r^{\infty}\left(\frac{d\psi}{dy}\right)^2 - \sinh^2\psi(y)\,dy\right)$$

where

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = \frac{1}{2}\sinh(2\psi), \;\; \psi(r) \sim \frac{2}{\pi}K_0(r), \;\; r \to \infty.$$

$y = e^{-\psi}$ is a particular **Painlevé III** transcendent and K_0 is the modified Bessel function.

SATO, MIWA & JIMBO, 1977–1980

τ-functions and holonomic quantum fields

A class of field theories that include the scaling limit of the Ising model and for which the expression of correlation functions in terms of solutions to holonomic differential equations is a general feature.

These developments led Jimbo-Miwa-Môri-Sato to consider, in 1980, the *Fredholm determinant* and *Fredholm minors* of the operator whose kernel is the familiar **sine kernel**

$$\frac{1}{\pi}\,\frac{\sin \pi(x-y)}{x-y}$$

on the domain $\mathbb{J} = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n)$.
Their main interest was the *density matrix of the impenetrable Bose gas*, and only incidentally, random matrices.

For $\mathbb{J} = (0, s)$, the **JMMS result** is

$$\det\left(I - \lambda K_{\mathrm{sine}}\right) = \exp\left(-\int_0^{\pi s} \frac{\sigma(x;\lambda)}{x}\, dx\right)$$

where

$$(x\sigma'')^2 + 4(x\sigma' - \sigma)\left(x\sigma' - \sigma + (\sigma')^2\right) = 0$$

with boundary condition

$$\sigma(x,\lambda) = -\frac{\lambda}{\pi}x + \mathrm{O}(x^2), \;\; x \to 0.$$

- σ is expressible in terms of Painlevé V.
- OKAMOTO analyzed the τ-function associated to Painlevé equations.

- A simplified derivation of the JMMS equations by TW.
- Connections with quantum inverse scattering were developed by ITS, KOREPIN and others.

RMM with Unitary Symmetry

Many RMM with unitary symmetry come down to the evaluation of Fredholm determinants $\det(I - \lambda K)$ where K has kernel of the form

$$\frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y}\,\chi_{\mathbb{J}}(y)$$

where

$$\mathbb{J} = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n).$$

Examples:

- Sine kernel: $\varphi(x) = \sin \pi x$, $\psi(x) = \cos \pi x$.
- Airy kernel: $\varphi(x) = \mathrm{Ai}(x)$, $\psi(x) = \mathrm{Ai}'(x)$.
- Bessel kernel: $\varphi(x) = J_\alpha(\sqrt{x})$, $\psi(x) = x\varphi'(x)$.
- Hermite kernel: $\varphi(x) = (\frac{N}{2})^{1/4}\varphi_N(x), \psi(x) = (\frac{N}{2})^{1/4}\varphi_{N-1}(x)$ where $\varphi_k(x) = \text{harmonic oscillator wave fns.}$

A **general theory** of such Fredholm determinants was developed by TW in the 1990s under the additional hypothesis that

$$m(x)\frac{d}{dx}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} A(x) & B(x) \\ -C(x) & -A(x) \end{pmatrix}\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

where m, A, B and C are polynomials. For example, for the Airy kernel

$$m(x) = 1,\ A(x) = 0,\ B(x) = 1,\ C(x) = -x.$$

The **basic objects** of the theory are

$$Q_j(x;\mathbb{J}) = (I - K)^{-1}\, x^j\varphi(x),\ \ P_j(x;\mathbb{J}) = (I - K)^{-1}\, x^j\psi(x),$$

and

$$u_j = (Q_j, \varphi)\,, v_j = (P_j, \varphi)\,, \tilde{v}_j = (Q_j, \psi)\,, w_j = (P_j, \psi)$$

where $(\cdot, \cdot)$ denotes the inner product. The **independent variables** are the endpoints a_j and b_j making up $\mathbb{J}$.

There are **two types of differential equations**:

- Universal equations.
- Equations that depend upon m, A, B and C.

For $K = K_{\mathrm{Airy}}$ with $\mathbb{J} = (s, \infty)$ reduces to **Painlevé II**

$$\frac{d^2 q}{ds^2} = s\, q + 2q^3$$

satisfying the boundary condition

$$q(s) \sim \mathrm{Ai}(s) \text{ as } s \to \infty.$$

This is called the Hastings-McLeod solution of Painlevé II. This leads to the distribution of the largest eigenvalue in GUE in the *edge scaling limit*

$$\boxed{F_2(x) = \exp\left(-\int_x^\infty (x-y) q(y)^2\, dy\right)}$$

- Palmer and Harnad & Its have an **isomondromic deformation** approach to these type of kernels.
- Adler, Shiota, & van Moerbeke's Virasoro algebra approach gives equations for the resolvent kernel $R(s,s)$.
- Given the DE, e.g. P_{II}, one is faced with the **asymptotic analysis of the solutions** which involves finding **connection formulae**, e.g./ as $x \to -\infty$,

$$\log\det F_2(x) = -\frac{x^3}{12} - \frac{1}{8}\log x + \kappa + \mathrm{O}(x^{-3/2})$$

where

$$\kappa = \frac{1}{24}\log 2 + \zeta'(-1)$$

Remark: The first two terms follow from the connection formula for P_{II}—the constant κ was conjectured in 1994 and only proved in 2006.

RMM with Orthogonal Symmetry

The added difficulty with RMM with orthogonal symmetry is that the kernels are **matrix kernels**. For example, for finite N GOE the operator is

$$K_1 = \chi \begin{pmatrix} K_2 + \psi \otimes \varepsilon\varphi & K_2 D - \psi \otimes \varphi \\ \varepsilon K_2 - \varepsilon + \varepsilon\psi \otimes \varepsilon\varphi & K_2 + \varepsilon\varphi \otimes \psi \end{pmatrix} \chi$$

where

$$K_2 \doteq \sum_{n=0}^{N-1} \varphi_n(x)\varphi_n(y),$$

ε is the operator with kernel $\frac{1}{2}\mathrm{sgn}(x-y)$, D is the differentiation operator, and χ is the indicator function for the domain $\mathbb{J}$.
Notation: $A \otimes B \doteq A(x)B(y)$.

The idea of the proof in TW is to factor out the GUE part

$$(I - K_2\chi)$$

and through various determinant manipulations show that the **remaining part is a finite rank perturbation**. Thus one ends up with formulas like

$$\det(I - K_1) = \det(I - K_2\chi)\ \det\left(I - \sum_{j=1}^{k} \alpha_j \otimes \beta_j\right)$$

For the case $\mathbb{J} = (s, \infty)$, an asymptotic analysis shows that as $N \to \infty$ the distribution of the scaled largest eigenvalue in GOE is expressible in terms of the **same** P_{II} function appearing in GUE.

The resulting GOE and GSE largest eigenvalue distribution functions are

$$\boxed{\begin{aligned} F_1(x) &= \exp\left(-\frac{1}{2}\int_x^\infty q(y)\,dy\right)(F_2(x))^{1/2} \\ F_4(x) &= \cosh\left(\frac{1}{2}\int_x^\infty q(y)\,dy\right)(F_2(x))^{1/2} \end{aligned}}$$

where

$$F_2(x) = \exp\left(-\int_x^\infty (x-y)q(y)^2\,dy\right)$$

and q is the Hastings-McLeod solution of P_{II}.

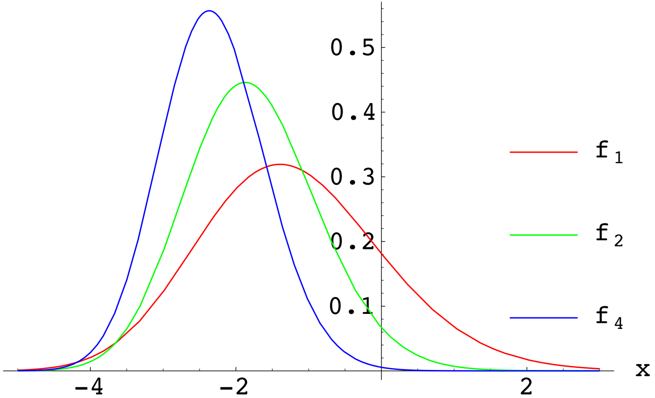

Figure: Largest eigenvalue densities $f_\beta(x) = dF_\beta/dx$, $\beta = 1, 2, 4$.

- The **edge scaling limit** is more **subtle for GOE** than for GUE or GSE. For GUE and GSE we have **convergence in trace norm** to limiting operators $K_{2,\mathrm{Airy}}$ and $K_{4,\mathrm{Airy}}$, but for GOE the **convergence is to a regularized determinant**.
- Ferrari & Spohn gave a different determinantal expression for F_1. It would be interesting to explore further their approach and its connection to the original GOE pfaffian approach of Dyson, et al.

- The asymptotics as $x \to -\infty$ is much more difficult and the complete solution was only recently achieved for $\beta = 1, 2, 4$ by Baik, Buckingham, DiFranco. As $x \to -\infty$

$$
\begin{aligned}
F_1(x) &= \tau_1 \frac{e^{-\frac{1}{24}|x|^3 - \frac{1}{3\sqrt{2}}|x|^{3/2}}}{|x|^{1/16}} \left(1 - \frac{1}{24\sqrt{2}|x|^{3/2}} + \mathrm{O}(|x|^{-3})\right), \\
F_2(x) &= \tau_2 \frac{e^{-\frac{1}{12}|x|^3}}{|x|^{1/8}} \left(1 + \frac{3}{2^6|x|^3} + \mathrm{O}(|x|^{-6})\right), \\
F_4(x) &= \tau_4 \frac{e^{-\frac{1}{24}|x|^3 + \frac{1}{3\sqrt{2}}|x|^{3/2}}}{|x|^{1/16}} \left(1 + \frac{1}{24\sqrt{2}|x|^{3/2}} + \mathrm{O}(|x|^{-3})\right)
\end{aligned}
$$

where

$$
\tau_1 = 2^{-11/48} e^{\frac{1}{2}\zeta'(-1)}, \ \ \tau_2 = 2^{1/24} e^{\zeta'(-1)}, \ \ \tau_4 = 2^{-35/48} e^{\frac{1}{2}\zeta'(-1)}
$$

- **Next-largest, etc. eigenvalue distributions** P_{II} type representations: unitary case TW; DIENG orthogonal and symplectic cases.

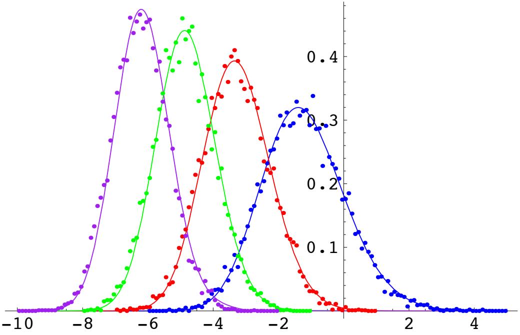

Figure: A histogram of the four largest (centered and normalized) eigenvalues for 10^4 realizations of $10^3 \times 10^3$ GOE matrices.

Universality Theorems

To what extent do the above limit laws depend upon the Gaussian and invariance assumptions for the probability measure?

Invariant Ensembles:
Replace Gaussian measure with

$$c_{N,\beta} \exp\left(-\beta \mathrm{tr}(V(A))/2\right)\, dA$$

where V is a polynomial of even degree and positive leading coefficient. This implies that the joint density for the eigenvalues is $(\beta = 1, 2, 4)$

$$P_{\beta,V,N}(x_1, \ldots, x_N) = C_{V,N,\beta} \prod_{1 \le i < j \le N} |x_i - x_j|^\beta \prod_{i=1}^N e^{-\beta V(x_i)/2}$$

Unitary ensembles ($\beta = 2$) are simpler than the orthogonal and symplectic ensembles ($\beta = 1, 4$), but both require for general V powerful RIEMANN-HILBERT METHODS for the asymptotic analysis.

Theorem. There exist constants $z_N^{(\beta)}$ and $s_N^{(\beta)}$ such that

$$\lim_{N\to\infty} \mathbb{P}_{\beta,V,N} \left(\frac{\lambda_{\max} - z_N^{(\beta)}}{s_N^{(\beta)}} \leq t \right) = F_\beta(t), \;\; \beta = 1, 2, 4,$$

Unitary case ($\beta = 2$): DEIFT, KRIECHERBAUR, MCLAUGHLIN, VENAKIDES AND ZHOU, and the orthogonal/symplectic: DEIFT & GIOEV. Special case $V(A) = \frac{1}{4}A^4 - gA^2$ BLEHER and ITS ($\beta = 2$) and STOJANOVIC ($\beta = 1$).

These deep theorems broadly extend the domain of attraction of the F_β limit laws.

Wigner Ensembles

Complex hermitian or real symmetric $N \times N$ matrices H

$$H = \frac{1}{\sqrt{N}} (A_{ij})_{i,j=1}^{N}$$

where A_{ij}, $1 \le i < j \le N$ are i.i.d. complex or real random variables with distribution μ. (Diagonal elements are i.i.d. real random variables independent of the off-diagonal elements.) The diagonal probability distribution is centered, independent of N and has finite variance.

Nongaussain Wigner ensembles define **non-invariant measures**.

No explicit formulas for the joint distribution of eigenvalues.

SOSHNIKOV proved, with μ symmetric (all odd moments are zero) and the distribution decays as at least as fast as a Gaussian distribution (together with a normalization on the variances):

Theorem.

$$\lim_{N\to\infty} \mathbb{P}_{W,N}\left(\lambda_{\max} \leq 1 + \frac{x}{2N^{2/3}}\right) = F_\beta(x)$$

with $\beta = 1$ for real symmetric matrices and $\beta = 2$ for complex hermitian matrices.

The importance of Soshnikov's theorem is the universality of F_β has been established for ensembles for which the "integrable" techniques, e.g. Fredholm theory, Riemann-Hilbert methods, Painlevé theory, are not directly applicable.

Edge Universality Theorems: Recent Developments

There are far reaching new results by two groups.

- Erdös, Yau & Yin: "The origin of the universality is due to the local ergodicity of Dyson Brownian motion." Eigenvalues of two generalized Wigner ensembles are equal in the large N limit provided that the second moments of the two ensembles are identical. This approach builds on some earlier work of Johansson. Ref: arXiv:1007.4652.
- Tao & Vu: A completely different approach. Ref: arXiv:0908.1982

Multivariate Statistical Analysis

Johnstone, 2006 ICM:

> *It is a striking feature of the classical theory of multivariate statistical analysis that most of the standard techniques—principal components, canonical correlations, multivariate analysis of variance (MANOVA), discriminant analysis and so forth—are founded on the eigenanalysis of covariance matrices.*

Thus it is not surprising that the methods of random matrix theory have important applications to multivariate statistical analysis.

Principal Component Analysis (PCA)

PCA with p variables have **population eigenvalues** ℓ_j, eigenvalues of the $p \times p$ covariance matrix

$$\Sigma = (\mathrm{Cov}(X_k, X_{k'}))_{1 \le k, k' \le p},$$

and **sample eigenvalues** $\hat{\ell}_j$, which are the (random) eigenvalues of the sample covariance matrix

$$S = \frac{1}{n} X X^T.$$

Here X is the $p \times n$ **data matrix** and

$$n = \text{number of observations of the } p \text{ variables.}$$

Since the parameters of the underlying probability model describing the random variables $X_1, \ldots, X_p$ are unknown, the problem is to deduce properties of Σ from the observed sample covariance matrix S.
Assume

$$\mathbb{X} = (X_1, \ldots, X_p)$$

is a p-variate Gaussian distribution $N_p(0, \Sigma)$ and the data matrix X is formed by n independent draws $\mathbb{X}_1, \ldots, \mathbb{X}_n$.
The $p \times p$ matrix XX^T is said to have p-variate **Wishart distribution** on n degrees of freedom, $\mathbf{W_p(n, \Sigma)}$.
Joint distribution of the eigenvalues $\hat{l}_j$: Complicated by the fact it involves an integral over the orthogonal group $\mathbb{O}(p)$.

Testing the Null Hypothesis: H_0

H_0: no correlations amongst the p variables, i.e. $\boldsymbol{\Sigma} = \mathbf{I}$.
Under H_0 all population eigenvalues $=1$, but there is a "spread" (Marčenko-Pastur) in the sample eigenvalues $\hat{\ell}_j$.
To assess whether "large" observed eigenvalues justify rejecting the null hypothesis, we need an approximation to the the **null hypothesis distribution** of the largest sample eigenvalue,

$$\mathbb{P}\left(\hat{\ell}_1 > t | H_0 = W_p(n, I)\right).$$

Theorem (Johnstone)

$$\mathbb{P}\left(n\hat{\ell}_1 \leq \mu_{np} + \sigma_{np} x | H_0\right) \longrightarrow F_1(x)$$

where $n \to \infty$, $p \to \infty$ such that $p/n \to \gamma \in (0, \infty)$,

$$\begin{aligned}
\mu_{np} &= \left(\sqrt{n - \frac{1}{2}} + \sqrt{p - \frac{1}{2}}\right)^2 \\
\sigma_{np} &= (\sqrt{n} + \sqrt{p}) \left(\frac{1}{\sqrt{n - \frac{1}{2}}} + \frac{1}{\sqrt{p - \frac{1}{2}}}\right)^{1/3}.
\end{aligned}$$

- Fractions $\frac{1}{2}$ in μ_{np} and σ_{np} improve the rate of convergence to F_1 to "second-order accuracy".

- Fractions $\frac{1}{2}$ in μ_{np} and σ_{np} improve the rate of convergence to F_1 to "second-order accuracy".
- El Karoui shows the theorem holds more generally as

$$p/n \to \gamma \in [0, \infty].$$

- Fractions $\frac{1}{2}$ in μ_{np} and σ_{np} improve the rate of convergence to F_1 to "second-order accuracy".
- El Karoui shows the theorem holds more generally as

$$p/n \to \gamma \in [0, \infty].$$

- For complex data matrices with $\Sigma = I$, there are corresponding limit theorems where now convergence is to F_2 (Johansson).

- Fractions $\frac{1}{2}$ in μ_{np} and σ_{np} improve the rate of convergence to F_1 to "second-order accuracy".
- El Karoui shows the theorem holds more generally as

$$p/n \to \gamma \in [0, \infty].$$

- For complex data matrices with $\Sigma = I$, there are corresponding limit theorems where now convergence is to F_2 (Johansson).
- Soshnikov and Péché **removed the assumption of Gaussian samples**. They assume that the matrix elements X_{ij} of the data matrix X are independent random variables with a common symmetric distribution whose moments grow not faster than the Gaussian ones.

- Fractions $\frac{1}{2}$ in μ_{np} and σ_{np} improve the rate of convergence to F_1 to "second-order accuracy".
- El Karoui shows the theorem holds more generally as

$$p/n \to \gamma \in [0, \infty].$$

- For complex data matrices with $\Sigma = I$, there are corresponding limit theorems where now convergence is to F_2 (Johansson).
- Soshnikov and Péché **removed the assumption of Gaussian samples**. They assume that the matrix elements X_{ij} of the data matrix X are independent random variables with a common symmetric distribution whose moments grow not faster than the Gaussian ones.
- **To summarize**, given the centering and norming constants and together with tables for F_1, one has a **good approximation to the null distribution**.

Spiked Populations: BBP Phase Transition

Case of **complex Wishart matrices**, $\mathbf{\Sigma} \neq \mathbf{I}$: Baik, Ben Arous & Péché for *complex* Wishart ensemble, with covariance matrix

$$\Sigma = \operatorname{diag}(\ell_1, \ldots, \ell_r, 1, \ldots, 1).$$

Consider $r = 1$ with $\ell_1 > 1$ and limit

$$p \to \infty, \; n \to \infty \text{ such that } \frac{p}{n} \to \gamma \geq 1.$$

Define

$$w_c = 1 + \sqrt{\gamma} \text{ and } \Phi(x) \text{ standard normal.}$$

Theorem. With Σ as above ($r = 1$), let $\hat{\ell}_1$ the largest eigenvalue of the sample covariance matrix.

- If $1 \leq \ell_1 < w_c$,

$$\mathbb{P}\left(\frac{n^{2/3}}{\sigma}\left(\hat{\ell}_1 - \mu\right) \leq x\right) \longrightarrow F_2(x),$$

$$\mu = (1+\sqrt{\gamma})^2, \;\; \sigma = (1+\sqrt{\gamma})(1+\frac{1}{\sqrt{\gamma}})^{1/3}.$$

Theorem. With Σ as above ($r = 1$), let $\hat{\ell}_1$ the largest eigenvalue of the sample covariance matrix.

- If $1 \leq \ell_1 < w_c$,

$$\mathbb{P}\left(\frac{n^{2/3}}{\sigma}\left(\hat{\ell}_1 - \mu\right) \leq x\right) \longrightarrow F_2(x),$$

$$\mu = (1 + \sqrt{\gamma})^2, \;\; \sigma = (1 + \sqrt{\gamma})(1 + \frac{1}{\sqrt{\gamma}})^{1/3}.$$

- If $\ell_1 > w_c$, then

$$\mathbb{P}\left(\frac{n^{1/2}}{\sigma_1}\left(\hat{\ell}_1 - \mu_1)\right) \leq x\right) \longrightarrow \Phi(x),$$

$$\mu_1 = \ell_1\left(1 + \frac{\gamma}{\ell_1 - 1}\right), \;\; \sigma_1 = \ell_1^2\left(1 - \frac{\gamma}{(\ell_1 - 1)^2}\right).$$

Remarks:

- The BBP theorem "*shows that a single eigenvalue of the true covariance Σ may drastically change the limiting behavior of the largest eigenvalue of sample covariance matrices. One should understand the above result as the statement that the eigenvalues exiting the support of the Marčenko-Pastur distribution form a small bulk of eigenvalues. This small bulk exhibits the same eigenvalue statistics as the eigenvalues of a non-normalized GUE (resp. GOE) matrix*".

Remarks:

- The BBP theorem "*shows that a single eigenvalue of the true covariance Σ may drastically change the limiting behavior of the largest eigenvalue of sample covariance matrices. One should understand the above result as the statement that the eigenvalues exiting the support of the Marčenko-Pastur distribution form a small bulk of eigenvalues. This small bulk exhibits the same eigenvalue statistics as the eigenvalues of a non-normalized GUE (resp. GOE) matrix*".
- If $\ell_1 = w_c$ the limiting distribution is a generalization of F_2 expressible in terms of the same Painlevé II function q.

Remarks:

- The BBP theorem "*shows that a single eigenvalue of the true covariance Σ may drastically change the limiting behavior of the largest eigenvalue of sample covariance matrices. One should understand the above result as the statement that the eigenvalues exiting the support of the Marčenko-Pastur distribution form a small bulk of eigenvalues. This small bulk exhibits the same eigenvalue statistics as the eigenvalues of a non-normalized GUE (resp. GOE) matrix*".
- If $\ell_1 = w_c$ the limiting distribution is a generalization of F_2 expressible in terms of the same Painlevé II function q.
- Patterson, Price & Reich have applied these results to problems of population structure arising from genetic data.

Remarks:

- The BBP theorem "*shows that a single eigenvalue of the true covariance* Σ *may drastically change the limiting behavior of the largest eigenvalue of sample covariance matrices. One should understand the above result as the statement that the eigenvalues exiting the support of the Marčenko-Pastur distribution form a small bulk of eigenvalues. This small bulk exhibits the same eigenvalue statistics as the eigenvalues of a non-normalized GUE (resp. GOE) matrix*".
- If $\ell_1 = w_c$ the limiting distribution is a generalization of F_2 expressible in terms of the same Painlevé II function q.
- PATTERSON, PRICE & REICH have applied these results to problems of population structure arising from genetic data.
- We mention that these same distributions play an analogous role in **canonical correlations** as they do in PCA.

Final Remarks

We have not discussed the appearance of the F_β limit laws in **growth processes**. This started with BAIK, DEIFT & JOHANSSON's work on **Ulam's Problem** of the length of the longest increasing subsequence of a random permutation.

Nor have we discussed the generalization of F_β to all real $\beta > 0$ by RAMÍREZ, RIDER & VIRÁG.